Linear transformation distance for bichromatic matchings*

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Abstract

Let $P = B \cup R$ be a set of $2n$ points in general position, where B is a set of n blue points and R a set of n red points. A *BR-matching* is a plane geometric perfect matching on P such that each edge has one red endpoint and one blue endpoint. Two BR-matchings are compatible if their union is also plane.

The *transformation graph of BR-matchings* contains one node for each BR-matching and an edge joining two such nodes if and only if the corresponding two BR-matchings are compatible. In SoCG 2013 it has been shown by Aloupis, Barba, Langerman, and Souvaine that this transformation graph is always connected, but its diameter remained an open question. In this paper we provide an alternative proof for the connectivity of the transformation graph and prove an upper bound of $2n$ for its diameter, which is asymptotically tight.

1 Introduction

A *geometric graph* $G(S, E)$ on a point set S in the plane is an embedding of a graph with the point set S as its vertex set and all edges embedded as straight line segments. $G(S, E)$ is called *plane* (or *crossing-free*) if no two of its edges share a point except for a possible common endpoint. A plane geometric graph is also called "planar straight-line graph" (PSLG for short). Two plane geometric graphs $G_1(S, E_1)$ and $G_2(S, E_2)$ on the same point set are called *compatible* if the union of their edge sets gives a plane geometric graph $G(S, E_1 \cup E_2)$, and *disjoint* if $E_1 \cap E_2$ is empty. Let P be a set of $2n$ points in the plane such that P does not contain three points on a common line, that is, P is in general position. A *plane geometric matching* on P is a plane geometric graph where each vertex is incident to at most one edge. In the following, we refer to plane geometric matchings just as *matchings*. A matching on P is called *perfect* if each vertex is incident to exactly one edge, that is, the number of edges in the matching is n.

The concept of matchings has a long history of research, so here we survey only briefly some of the most recent results. Sharir and Welzl [15] provided bounds on the number of perfect matchings, all matchings (not necessarily perfect), and other variations of matchings that exist on a set P. Aichholzer et al. [1] formulated the *Disjoint Compatible Matching Conjecture* which was then proved by Ishaque et al. [9]: For every perfect matching with an even number of edges there exists a disjoint compatible perfect matching. In a slightly different direction, the compatibility of perfect matchings and different classes of plane geometric graphs is investigated. In [2] it is shown that for outerplanar graphs there always exists a compatible perfect matching. Further, upper and lower bounds are given on the number of edges shared between the given plane geometric graph and a compatible perfect matching, in case the graph is either a tree or a simple polygon.

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Let M and M' be two perfect matchings on P. According to [1] a *transformation of length k between M and M'* is a sequence of perfect matchings $M = M_0, \ldots, M_k = M'$ such that M_{i-1} and M_i are compatible for all $1 \leq i \leq k$. Let the *transformation graph (of perfect matchings on P)* be the graph containing one node for each perfect matching on P and an edge joining two such nodes if and only if the corresponding two perfect matchings are compatible, that is, there exists a transformation of length 1 between these two perfect matchings. Aichholzer et al. [1] proved that there always exists a transformation of length $O(\log n)$ between any two matchings of P. Hence, the transformation graph is connected with diameter $O(\log n)$. Providing a lower bound for the diameter, Razen [14] proved that there exist point sets P such that the transformation graph (of P) has diameter $\Omega(\log n / \log \log n)$.

Given the wide interest in work on bichromatic point sets (see [10] for a survey) it is only natural to extend the questions on matchings into that direction. For the rest of this paper let $P = B \cup R$ be a bichromatic set of $2n$ points in the plane in general position, where $|B| = |R| = n$. We call B the set of blue points and R the set of red points. An edge of a geometric graph on P is called *bichromatic* if one endpoint of the edge is in B and the other endpoint is in R. A geometric graph is bichromatic, if all its edges are bichromatic. For brevity, and in accordance with [4], a perfect matching M on P is termed a *BR-matching* if M is bichromatic, that is, all edges of M are bichromatic.

It is well known that a BR-matching always exists for any set P as defined above. For proofs see, e.g., [11, p. 51] (using the "minimum weight is plane" argument) and [11, pp. 200–201] (using the intermediate value theorem). On every set P there also always exists a BR-matching constructed by repeated application of a "ham-sandwich cut" (see Figure 1). We use such a BR-matching as the canonical structure (following the lines of [4]) and thus describe this in more detail in Section 2. Concerning the maximal number of BR-matchings (over all sets P with $|P| = 2n$), Sharir and Welzl [15] proved that it is at most $O(7.61^{2n})$ and can be bounded from below by $\Omega(2.23^{2n}/\operatorname{poly}(n))$ (where $\operatorname{poly}(n)$ stands for a polynomial factor in n).

In a different direction, the augmentation of a disconnected bichromatic plane geometric graph with no isolated vertices to a connected bichromatic plane geometric graph has been considered. The resulting connected (bichromatic) plane geometric graph is often called "(bichromatic) encompassing graph". Hurtado et al. [8] proved that such an augmentation is always possible and provided an $O(n \log n)$ time algorithm to construct one. This implies as a special case that every BR-matching can be augmented to a bichromatic plane spanning tree in $O(n \log n)$ time. The result was extended by Hoffmann and Tóth [7] to augmenting bichromatic geometric plane graphs to bichromatic encompassing graphs where the increase of the degree of each vertex during the augmentation is bounded by two. Thus, any BR-matching can be augmented to a bichromatic plane spanning tree with bounded degree three. In a similar line of research Aichholzer et al. [3] proved that for every BR-matching there exists a bichromatic disjoint compatible matching M' on P with at least $\lceil \frac{n-1}{2} \rceil$ edges. Furthermore, for an upper bound they provided an example where M' has at most $3n/4$ edges.

Let M and M' be two BR-matchings. Similar to the uncolored setting, a *transformation of length k between M and M'* is a sequence of BR-matchings $M = M_0, \ldots, M_k = M'$ such that M_{i-1} and M_i are compatible for all $1 \leq i \leq k$. The *transformation graph $\mathcal{M}_{BR}$* (of BR-matchings) is the graph containing one node for each BR-matching and an edge joining two such nodes if and only if the corresponding two BR-matchings are compatible. Aloupis et al. [4] recently answered a question posed in [3], proving that $\mathcal{M}_{BR}$ is connected for every point set $P = B \cup R$. They presented a linear lower bound example for the maximum of the diameter of $\mathcal{M}_{BR}$ over all P. However, they provided no upper bound other than the trivial exponential bound stemming from the maximal number of nodes of $\mathcal{M}_{BR}$.

By adapting the approach and some of the tools presented in [4] we give an alternative proof of the connectivity of $\mathcal{M}_{BR}$. A detailed analysis of each step of this proof allows us to prove an upper bound of $2n$ for the diameter of $\mathcal{M}_{BR}$. This is asymptotically tight, as there exist point sets P for which $\mathcal{M}_{BR}$ has diameter $n/2$ (see [4] and Figure 2).

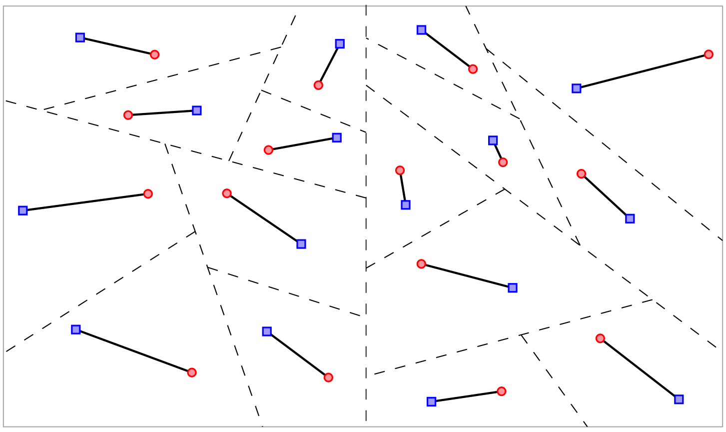

Figure 1: [4] A ham-sandwich matching obtained by repeated application of ham-sandwich cuts. In our figures we depict blue points as filled squares and red points as filled disks.

2 The main result

The main result of this paper is an asymptotically tight upper bound on the diameter of the transformation graph $\mathcal{M}_{BR}$ of BR-matchings, derived by an alternative proof of the connectivity of $\mathcal{M}_{BR}$. To this end, we define a canonical BR-matching and show that there exists a transformation of linear length between any BR-matching and the canonical one.

Throughout this paper, a *ham-sandwich cut* of P is a straight line ℓ such that (1) exactly $\lfloor \frac{n}{2} \rfloor$ blue and $\lfloor \frac{n}{2} \rfloor$ red points of P are on one side of ℓ and (2) exactly $\lceil \frac{n}{2} \rceil$ blue and $\lceil \frac{n}{2} \rceil$ red points of P are on the other side of ℓ, which implies that ℓ does not contain any point of P. (Recall that we assume general position on P.) For even n this definition matches the "classical" definition for a ham-sandwich cut. By the so-called *Ham-sandwich Theorem* such a ham-sandwich cut always exists. See [5], [6], [12], and [13, Chapter 3] for detailed information. Furthermore, it is known that a ham-sandwich cut can be computed in $O(n)$ time [12]. For odd n a "classical" ham-sandwich cut ℓ_c of P would contain a red and a blue point (on ℓ_c). We can shift ℓ_c slightly in parallel to achieve a ham-sandwich cut as defined above.

We construct a BR-matching H by recursively applying ham-sandwich cuts until in any cell there remain only two points, one of each color, which are then matched (see Figure 1). Recall that this is always possible by the Ham-sandwich Theorem. In accordance with [4] we call H a *ham-sandwich matching*. Note that several different ham-sandwich matchings might exist on P and that, in general, not every BR-matching is a ham-sandwich matching. Further, there exist point sets P that admit only one single BR-matching, which then is a ham-sandwich matching.

One important ingredient for proving our main result (Theorem 2.2) is Lemma 2.1 stated below. A similar result was obtained in [4] using comparable methods. However, that result did not permit to prove an upper bound on the diameter of $\mathcal{M}_{BR}$ (other than the trivial exponential one). To not disrupt the train of thought we defer the proof of Lemma 2.1 to Section 3.4, as the remainder of this paper provides the tools for this proof.

Two BR-matchings M and M' are said to be *t-compatible* if there exists a transformation of length k between M and M', with $k \leq t$.

Lemma 2.1. *Let $P = B \cup R$ be a bichromatic set of $2n$ points in the plane in general position such that $|B| = |R| = n$. For every BR-matching M and every ham-sandwich cut ℓ of P, there exists a BR-matching M^ℓ such that M and M^ℓ are $\lfloor n/2 \rfloor$-compatible and no edge of M^ℓ intersects ℓ.*

Using this lemma, we obtain our main result.

Theorem 2.2. *Let $P = B \cup R$ be a bichromatic set of $2n$ points in the plane in general position such that $|B| = |R| = n$. For every BR-matching M and every ham-sandwich matching H of P, M and H are n-compatible.*

Proof. We prove the statement by induction on n. Trivially, the claim is true for $n = 1$. Hence, we proceed with the induction step and assume that the claim is true for any $1 \leq n' < n$.

Let ℓ be the first ham-sandwich cut in the construction of H, i.e., a ham-sandwich cut of P. By Lemma 2.1, there is a BR-matching M^ℓ such that M and M^ℓ are $\lfloor n/2 \rfloor$-compatible and no edge of M^ℓ intersects ℓ. Let $P_1 = B_1 \cup R_1$ and $P_2 = B_2 \cup R_2$ be the subsets of points of P lying to the left and to the right of ℓ, respectively. For each $i \in \{1, 2\}$, let M_i^ℓ and H_i be the subgraphs of M^ℓ and H, respectively, which are induced by P_i. (Note that $H_1 \cup H_2 = H$ and $M_1^\ell \cup M_2^\ell = M^\ell$ as no edges of M^ℓ and H intersect ℓ.)

Let ℓ_1 and ℓ_2 be the ham-sandwich cuts of P_1 and P_2, respectively, used to construct H. Because $|P_i| = 2n' \leq 2\lceil n/2 \rceil < 2n$, M_i^ℓ and H_i are $\lceil n/2 \rceil$-compatible by induction. Moreover, observe that every B_1R_1-matching is compatible with (and disjoint from) every B_2R_2-matching. Thus, the two transformations of length k_i between M_i^ℓ and H_i ($k_i \leq \lceil n/2 \rceil$) can be "merged" (i.e., executed in parallel) to one transformation of length $\max_i\{k_i\}$ between M^ℓ and H. Finally, as M and M^ℓ are $\lfloor n/2 \rfloor$-compatible and M^ℓ and H are $\lceil n/2 \rceil$-compatible, we conclude that M and H are n-compatible. □

Corollary 2.3. *Let $P = B \cup R$ be a bichromatic set of $2n$ points in the plane in general position such that $|B| = |R| = n$. The transformation graph $\mathcal{M}_{BR}$ is connected with diameter at most $2n$.*

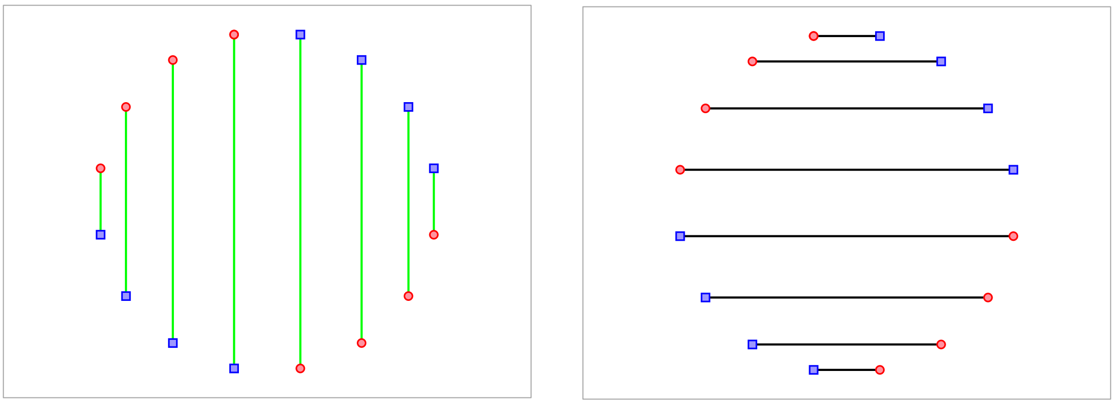

Figure 2: [4] Two ham-sandwich matchings that have distance $n/2$ in the transformation graph.

The example depicted in Figure 2, which has also been presented in [4], shows that the diameter of the transformation graph $\mathcal{M}_{BR}$ can be as high as $n/2$. Together with Corollary 2.3, we obtain the following result.

Corollary 2.4. *The maximum over all bichromatic sets $P = B \cup R$ with $|B| = |R| = n$ of the diameter of the transformation graph $\mathcal{M}_{BR}$ is $\Theta(n)$.*

Note that the lower bound for the diameter of $\mathcal{M}_{BR}$ is 0, as there exist point sets $P = B \cup R$ with $|B| = |R| = n$ admitting only one BR-matching.

3 Proof of Lemma 2.1

For the remainder of this paper, we consider each edge of a plane geometric graph G to have two sides. Formally, each edge pq of G consists of a pair of *half-edges*, one directed from p to q and the other directed from q to p such that the cycle of each half-edge pair is oriented clockwise (see Figure 3 (a)). Each half-edge is colored either red or blue. For an edge pq the half-edge directed to p is called the *twin* of the half-edge directed to q, and vice versa. Let ℓ_{pq} be the line supporting the edge pq, and being directed from p to q. Only the half-edge directed to q is visible from the left side of ℓ_{pq} whereas only the half-edge directed to p is visible from the right side of ℓ_{pq}. In other words, a half-edge is visible only from its left side and has its twin on its right side. Note that a point x on an open edge pq with differently colored half-edges is observed as being red from one side of ℓ_{pq}, while x appears to be blue from the other side of ℓ_{pq} (see again Figure 3 (a)).

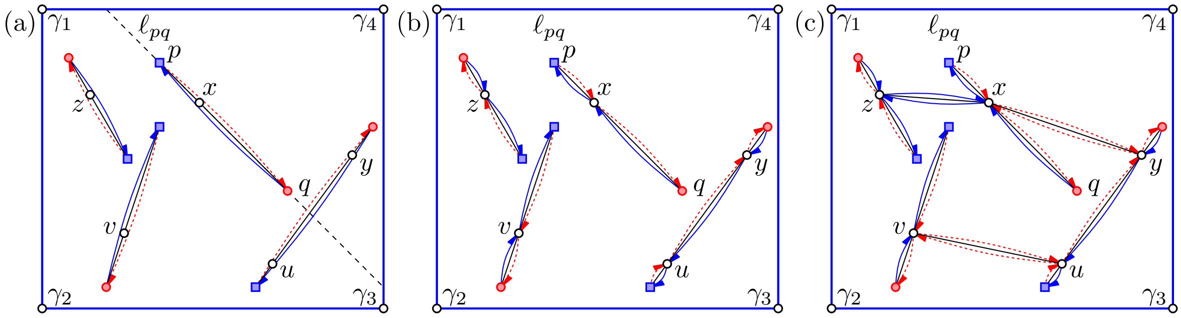

Figure 3: Splitting and gluing in P-graphs: In the figures we show points of Q as white disks, blue half-edges as solid arcs, and red half-edges as dotted arcs. The points $\{\gamma_1, \ldots, \gamma_4\}$ are the vertices of the rectangle Γ. For simplicity Γ is displayed with bold lines instead of equally-colored half-edges. (a) Each edge pq of M has two half-edges, one half-edge directed to p and colored like p, the other directed to q and colored like q. The points z and u are not visible, x and u are visible but not color-visible, and x and y are color-visible. (b) The resulting graph when splitting pq at x and the other edges at z, v, u, and y. (c) The resulting graph after gluing three pairs of color-visible points x, y, x, z, and u, v.

Let M be a BR-matching. For each edge s of M color the half-edges of s in the same color as the endpoint towards which they are directed to. In this way, every edge of a BR-matching has a blue half-edge and a red half-edge. Moreover, this coloring is uniquely determined by P (and the fixed orientation of half-edge pairs). Let Γ be an axis aligned rectangle sufficiently large to enclose M in its interior. We color each half-edge on the boundary of Γ with the same color (to be determined later). See Figure 3 (a) for an illustration where each half-edge of Γ is colored blue.

We define a *P-graph* (of M and Γ) to be a plane geometric graph G_M on a point set $P \cup Q$ such that (1) Q is disjoint from P, (2) G_M contains a subdivision of Γ and a subdivision of M as subgraphs, (3) for every edge of M its half-edges are colored as defined above, and (4) for every edge of G_M that is not an edge of M, its two half-edges are colored in the same color, either red or blue. (We do not require $P \cup Q$ to be in general position, but recall that we assume general position of P.) From now on we only consider the part of the plane bounded by Γ. Thus, each considered face f of G_M is bounded. We denote by ∂f the boundary of f and by $\text{int}(f)$ the interior of f. Furthermore, let the *boundary* of G_M, denoted by ∂G_M, be the union of all the edges in G_M, and let the *interior* of G_M be the union of the interiors of its faces.

Consider two points x and y that lie on different edges of ∂G_M. We say that x and y are *visible* if the open segment joining x with y is contained in the interior of G_M. We say that x and y are *color-visible* if they are visible and the color of x when viewed from y is equal to the color of y when viewed from x. For example, in Figure 3 (a), u and x are visible but not color-visible, while x and y are color-visible.

With these definitions, we first show how to create a P-graph of M that is a convex decomposition of the interior of Γ. To this end we define the glue operation, as has been done in [4], and use a colored version of an extension of a matching (see e.g. [1] for uncolored extension). Then we show how to construct a BR-matching that is compatible to the created convex decomposition and prove that this BR-matching has strictly less intersections with a ham-sandwich cut of P than M.

3.1 Splitting and gluing in P-graphs

Consider a P-graph G_M on $P \cup Q$ and let $x \notin P \cup Q$ be a point on an edge pq of G_M. To *split* pq at x we do the following: (1) add x to Q, (2) add the edges px and xq to G_M, (3) color the half-edges from p to x and from x to q like the half-edge from p to q, and the other two new half-edges like the half-edge from q to p, and (4) remove pq (and its two half-edges) from G_M. Figure 3 (a-b) gives an illustration of the split operation.

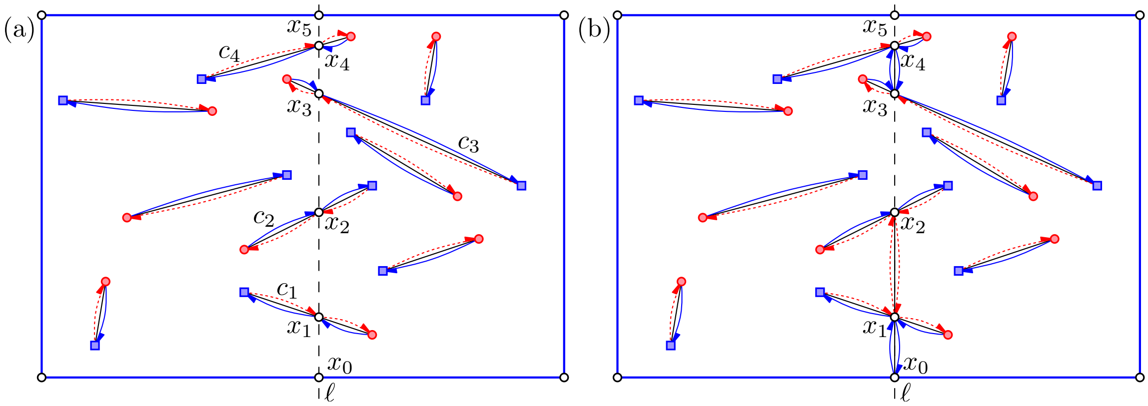

Figure 4: Generating G_M^0: (a) The edges $\langle c_1, \ldots, c_4 \rangle$ of M intersect ℓ in $\langle x_1, \ldots, x_4 \rangle$ and are split at these points. (b) The pairs of color-visible points x_0, x_1, x_1, x_2, and x_3, x_4 are glued.

We borrow the gluing technique introduced in [4]: Let y and y' be two color-visible points on two different edges e and e', respectively, of ∂G_M such that neither y nor y' is in P. To *glue* y with y', we do the following: If y (or y') is not a vertex of G_M, then we split e at y (or e' at y'), by this ensuring that y and y' are now vertices of G_M. Then we add the edge yy' to G_M and color the two half-edges of yy' with the same color as y when viewed from y'. See Figure 3 (b-c) for examples of gluing.

Observation 3.1. *The resulting graph of splitting an edge of a P-graph at a point on this edge is again a P-graph. The resulting graph of gluing two color-visible points (neither of them in P) on two different edges of a P-graph is again a P-graph.*

Consider a P-graph G_M on $P \cup Q$ with Q only containing the four points of Γ and G_M containing only the edges of M and Γ. Let ℓ be a ham-sandwich cut of P and assume without loss of generality that ℓ is vertical and that no edge of M is parallel to ℓ. Let $C_{M,\ell} = \langle c_1, \ldots, c_k \rangle$ be the sequence of k edges of M that intersect ℓ, sorted from bottom to top according to the point of intersection x_i of c_i with ℓ. Let x_0 and x_{k+1} be the intersection points of ℓ with the bottom edge and top edge of Γ, respectively. Color each half-edge on the boundary of Γ with the same color as x_1 when viewed from x_0; see Figure 4 (a). Recall that Lemma 2.1 looks for a BR-matching M^ℓ, such that M and M^ℓ are compatible and M^ℓ has no edges intersecting ℓ. Therefore, we can assume that $k > 0$ as otherwise we have already found the desired BR-matching. We construct a P-graph G_M^0 by gluing x_i with x_{i+1}, for each $0 \leq i \leq k$, if x_i and x_{i+1} are color-visible. By doing so, we ensure that no edge in a BR-matching compatible with G_M^0 can intersect ℓ between x_i and x_{i+1}, if x_i and x_{i+1} are color-visible. Recall that the half-edges on Γ have the color of c_1 when viewed from below. That is, the points x_0 and x_1 are color-visible and hence, they are glued together; see Figure 4 (b) for an illustration.

Observation 3.2. *Let M be any BR-matching on P and let ℓ be any ham-sandwich cut of P, such that the intersection of ℓ with the edges of M is not empty. There exists a P-graph G_M^0 such that two points x_i and x_{i+1} are joined by an edge in G_M^0 if and only if x_i and x_{i+1} are color-visible. Moreover, x_0 and x_1 are always glued by an edge of G_M^0.*

3.2 Extension of M

In this section, we describe the extension of the BR-matching M in the P-graph G_M^0. Let $s_1, \ldots, s_n$ be an arbitrary order of the edges of M. Starting with G_M^0, we extend each edge of M in this order, resulting in a sequence $G_M^0, \ldots, G_M^n$ of P-graphs.

During this sequence we maintain the following *color-invariant*: For $0 \leq j \leq n$ and every pair of points $u, v \in (\partial G_M^j \cap \ell)$, u and v are not color-visible. Intuitively, the color-invariant guarantees

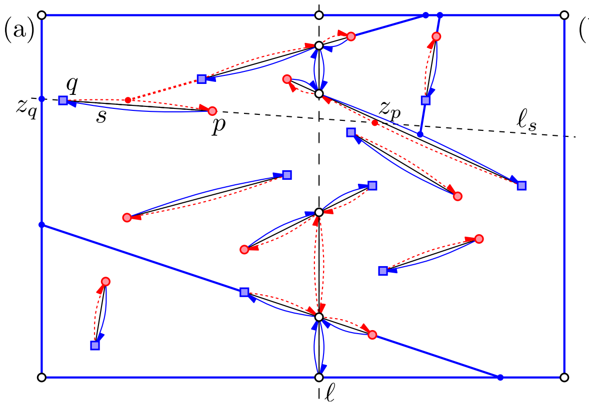

Figure 5: Extending M in G_M^0: For simplification, extensions of edges are displayed with bold lines instead of equally-colored half-edges and endpoints of extensions are depicted as small dots. (a) The edge $s = pq$ is extended on its supporting line ℓ_s, hitting the blue point z_q and the red point z_p. (b) As z_q is blue and z_p is red, the extensions qz_q and pz_p are blue and red, respectively. The extension pz_p intersects ℓ in the point y inside the interval $z_d z_u$ on ℓ. As y and z_u are color-visible (red), they are glued.

that every interval along ℓ that is not covered by an edge of G_M^j is bounded by points having different colors.

Lemma 3.3. *The color-invariant holds for G_M^0.*

Proof. Recall that x_0 and x_{k+1} are the intersections of ℓ with Γ and that for every $1 \leq i \leq k$, x_i is the intersection of the edge $c_i \in C_{M,\ell}$ with the line ℓ. For two points to be color-visible they need to be visible. In $\partial G_M^0 \cap \ell$ only the points x_i and x_{i+1}, for some $0 \leq i \leq k$, can be visible. By Observation 3.2, x_i and x_{i+1} are visible in G_M^0 if and only if they are not color-visible in M, i.e., the color-invariant holds. □

We proceed by describing the extension of M in detail. For each edge s_j of M the *extension* of s_j comes in three steps: (1) shooting a ray from s_j to both directions until hitting an edge of G_M^{j-1}, (2) proper coloring of the half-edges of the two rays, and (3) maintaining the color-invariant.
Step 1: Let ℓ_s be the supporting line of $s_j = pq$. Let z_p and z_q be the intersection points of ℓ_s and ∂G_M^{j-1}, such that p and z_p are visible and q and z_q are visible. Note that such an intersection can be with an edge of M, with an edge of Γ, or with any other edge of ∂G_M^{j-1}. If any of z_p or z_q is not a vertex of G_M^{j-1} then split the edge containing z_p at z_p or split the edge containing z_q at z_q, respectively. Extend s_j by adding the edges pz_p and qz_q to G_M^{j-1}. See Figure 5 (a) for an example.
Step 2: The two half-edges of pz_p are colored with the same color as z_p when viewed from p. The two half-edges of qz_q are colored with the same color as z_q when viewed from q; see Figure 5 (b). By this coloring, the resulting graph is a P-graph.
Step 3: Observe that at most one of the two new edges can intersect ℓ. Assume that the color-invariant holds before processing s_j. If neither of the two new edges intersects ℓ, then the color-invariant still holds after extending s_j. Thus, without loss of generality, assume that pz_p intersects ℓ in point y. Let z_u and z_d be first points hit on ∂G_M^{j-1} when shooting upwards and downwards, respectively, from y along ℓ. The color-invariant guarantees that z_u and z_d are not color-visible in G_M^{j-1}. Hence, z_u and z_d have different colors when viewed from y, but y has the same color independent of being viewed from z_u or z_d. Therefore, y and exactly one of the two points z_u and z_d are color-visible. We glue y with this color-visible point; see Figure 5 (b) for an example.

Lemma 3.4. *The color-invariant is preserved after each extension of an edge of M. In particular, this invariant holds in the resulting graph G_M^n, after extending every edge of M.*

Proof. By Lemma 3.3, the color-invariant holds for G_M^0 before extending s_1. We prove by induction and thus assume that the color-invariant is preserved until extending s_j. Observe that the color-invariant can only be violated if a new edge (at most one of the two extensions of s_j) intersects ℓ in a point y. If this is the case then y lies between two points z_d and z_u that are visible in G_M^{j-1}. As argued above, y and exactly one of the two points, without loss of generality z_d, are color-visible in G_M^{j-1}. As y is glued with z_d in Step 3, y and z_d are not visible in G_M^j. Furthermore, all other pairs of visible points of ∂G_M^{j-1} on ℓ remain unchanged. Thus, the color-invariant also holds after extending s_j. $\square$

It is easy to see that the resulting P-graph G_M^n decomposes the interior of Γ into convex simple polygons, each being a face of G_M^n; see Figure 6 (a). Note that every point in the interior of each face of G_M^n sees a counterclockwise directed cycle of colored half-edges.

In the following two sections we construct a BR-matching M' compatible to G_M^n. Recall that the edges of M' should have as few intersections with the ham-sandwich cut ℓ as possible. As G_M^n and M' are compatible, only edges inside a face of G_M^n can intersect ℓ. Thus, we are interested in the number of faces of G_M^n that contain a portion of ℓ in their interior. We say that a face f of G *crosses* ℓ if $\mathrm{int}(f) \cap \ell \neq \emptyset$.

Lemma 3.5. *At most $k-1$ faces of G_M^n cross ℓ, where $k = |C_{M,\ell}|$.*

Proof. Recall that $C_{M,\ell} = \langle c_1, \ldots, c_k \rangle$ is the sequence of edges of M that intersect ℓ and that for every $1 \leq i \leq k$, x_i is the intersection point of c_i with ℓ. Further recall that x_0 and x_{k+1} are the intersections of ℓ with Γ and that we assume that $k > 0$, as otherwise M would already fulfill the requirements of Lemma 2.1. In [4] it was already observed that if ℓ intersects at least one edge of M, then it must intersect an even number of edges of M. Moreover, as ℓ is a ham-sandwich cut, at each side of ℓ the number of red points equals the number of blue points. Therefore, if we consider the endpoints of the edges in $C_{M,\ell}$ at one side of ℓ, half of them must be blue and half must be red. Otherwise, the numbers of remaining red and blue points at that side of ℓ would be unbalanced, leading to a contradiction with M being a BR-matching. Thus, there exists at least one $\xi \in \{1, \ldots, k-1\}$ such that the pair of consecutive edges c_ξ and $c_{\xi+1}$ in $C_{M,\ell}$ has differently colored endpoints at the same side of ℓ. By the coloring scheme of the half-edges of M, x_ξ and $x_{\xi+1}$ are color-visible in M.

For $0 \leq j \leq n$, let ω_j be the number of connected components of $\ell \setminus \partial G_M^j$ that lie inside Γ. Observe that inside Γ the number of connected components of ℓ intersected by the edges of M is $k+1$. By Observation 3.2, x_i is glued with x_{i+1} in the construction of G_M^0 if and only if x_i and x_{i+1} are color-visible. By the choice of the color of the half-edges of Γ, x_0 is glued with x_1. As argued above, there exists at least one additional pair x_ξ and $x_{\xi+1}$ that is color-visible and thus glued in G_M^0. Hence, ω_0 is at most $k-1$.

In the construction of G_M^j, $1 \leq j \leq n$, the connected components of $(\ell \setminus \partial G_M^{j-1}) \cap \Gamma$ remain unchanged unless exactly one new edge intersects ℓ. In this case, exactly one connected component gets split into two connected components, of which exactly one connected component is removed in G_M^j by gluing its endpoints. Thus, $\omega_j = \omega_{j-1}$ for all $1 \leq j \leq n$.

As the faces of G_M^n are convex simple polygons, the number of faces of G_M^n that cross ℓ is equal to ω_n and thus at most $k-1$. $\square$

3.3 Switch vertices and switch matchings

Recall that G_M^n decomposes the interior of Γ into convex faces. The idea is to assign each point of P to a unique face of this decomposition, such that every face has a balanced number of (possibly zero) red and blue points assigned. This way, we obtain a new BR-matching by independently matching the points assigned to each face of this decomposition.

Note that each half-edge of G_M^n is incident to the interior of a unique face f of G_M^n. Therefore, we can think of ∂f to be composed of all the half-edges incident to $\mathrm{int}(f)$. Consider the sequence $h_0, \ldots, h_{t-1}$ of the $t \geq 3$ half-edges along ∂f in counterclockwise order, i.e., the cycle formed of the t half-edges incident to $\mathrm{int}(f)$.

A vertex v of G_M^n is a *switch-vertex* in f if the two half-edges h_i and h_{i+1} (with $i \in \{0 \ldots t-1\}$ and indices taken modulo t) that are incident to $\mathrm{int}(f)$ and adjacent to v have different colors; see

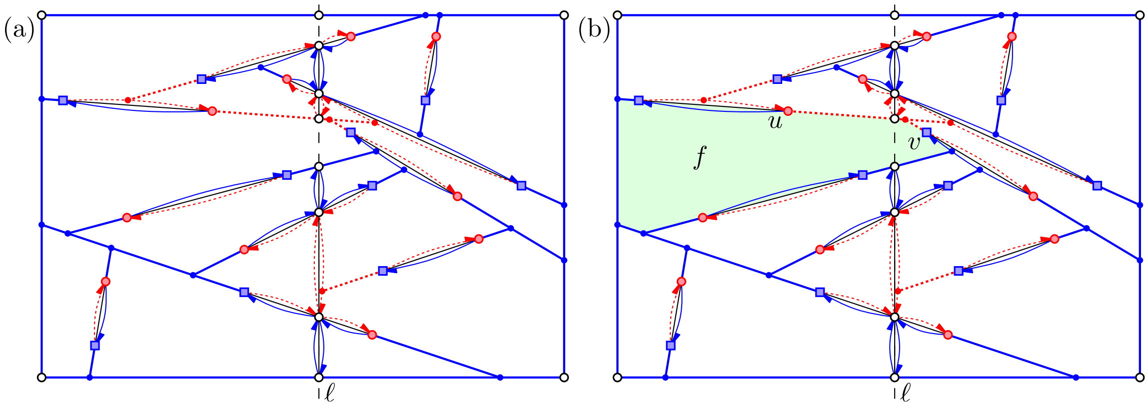

Figure 6: (a) The convex decomposition of Γ obtained after extending every edge of M. (b) The vertices u and v are the only switch-vertices in the face f.

Figure 6 (b) for an illustration. In other words, v is switch-vertex in some face if, in the cyclic order of incident half-edges around v, two consecutive half-edges that are not twins have different color.

Lemma 3.6. *A vertex of G_M^n is a switch-vertex in one of its faces if and only if it is a point of P. Furthermore, a vertex can be a switch-vertex in at most one face of G_M^n.*

Proof. For each point q of G_M^n let Δ_q be the cyclic order of its incident half-edges.

First observe that splitting an edge of a P-graph at a point x preserves Δ_q for all $q \in P \cup Q \setminus \{x\}$. (Strictly speaking, at the endpoints of the split edge the split half-edges get exchanged with the new half-edges. But as their color stays the same, the cyclic order of the colors of half-edges around these points stays the same.) Further, for the new point $x \in Q$, Δ_x contains two pairs of consecutive half-edges that are not twins, and both pairs consist of equally-colored half-edges. Hence, the split operation preserves existing switch-vertices and does not create new ones.

Second, let $y \in Q$ be a point that is glued with another point in Q. This means that two equally-colored half-edges are inserted between two equally-colored half-edges of the same color in Δ_y. Therefore, no point in Q becomes a switch-vertex by the glue operation.

Third, let z be the point on some edge of the P-graph that is first hit by the extension of one side of some edge of M. If not already in Q, z gets added to Q by a split operation. Then, like in the glue operation, two equally-colored half-edges are inserted between two equally-colored half-edges of the same color in Δ_z. Again, no point in Q becomes a switch-vertex by this operation.

Altogether, no point of Q is turned into a switch-vertex during the construction of G_M^n. Further, all points in Q are either points of Γ (whose incident half-edges are all of the same color) or created in a split operation. Therefore, no point in Q is a switch-vertex.

Concerning the set P recall that each point $p \in P$ is an endpoint of an edge $s = pp'$ of M. As argued above, only the extension of s alters Δ_p during the construction of G_M^n. Before the extension of s, each of the endpoints p and p' of s is incident to exactly one twin pair of half-edges (where one is colored red and the other one is colored blue). The extension of s adds two additional half-edges to p, both of the same color. Thus, Δ_p has exactly two pairs of consecutive half-edges that are not twins, and for exactly one of them the two half-edges differ in color. The same statement holds for $\Delta_{p'}$. Note that for all points $\tilde{p} \in P \setminus \{p, p'\}$ this operation preserves $\Delta_{\tilde{p}}$. Therefore, every point in P is a switch-vertex for exactly one face of G_M^n. □

Lemma 3.7. *Let $h_0, \ldots, h_{t-1}$ be the sequence of half-edges along the boundary of a face f of G_M^n in counterclockwise order. Let v_i be a switch-vertex in f and let h_i and h_{i+1} (indices taken modulo t) be the two half-edges incident to v_i. Then v_i has the same color as h_i while h_{i+1} is of the opposite color.*

Proof. By Lemma 3.6, v_i is a point of P. Hence, v_i is the endpoint of an edge s of M. Let s' be the part of s (after possible splits) incident to v_i in G_M^n. Recall that splitting an edge of a P-graph preserves the cyclic order of incident half-edges for all points in P. Therefore, the half-edge h^+ of s' directed towards v_i has the same color as v_i, and the half-edge h^- of s' directed away from v_i has the opposite color of v_i.

In case that h_i is h^+, h_i has the same color as v_i and, since v_i is a switch-vertex, h_{i+1} must be of the opposite color. In the other case, where h_{i+1} is h^-, h_{i+1} is of the opposite color as v_i and as v_i is a switch-vertex, h_i must have the same color as v_i. Thus, in both cases the claim in the lemma is true. □

We say that a face f of G_M^n is *well-colored* if the sequence of switch-vertices along ∂f alternates in color. Analogously, a P-graph is *well-colored* if all its faces are well-colored. Notice that if a face is well-colored, then it has an even number of switch-vertices.

Lemma 3.8. *Every face of G_M^n is well-colored.*

Proof. Let $h_0, \dots, h_{t-1}$ be the sequence of half-edges along the boundary of a face f of G_M^n in counterclockwise order. For any $0 \leq i \leq t-1$, let v_i be the vertex shared by h_i and h_{i+1} (indices taken modulo t). Recall that h_i and h_{i+1} have different colors if and only if v_i is a switch-vertex in f.

Let v_i and v_j be two consecutive switch-vertices along ∂f such that $i < j < t$. Assume without loss of generality that v_i is red. Therefore, Lemma 3.7 implies that h_i is red whereas h_{i+1} is blue. Because v_i and v_j are consecutive switch-vertices along ∂f, for every $i < r < j$, v_r is not a switch-vertex. Thus, $h_{i+1}, \dots, h_j$ share the same color, i.e., they are blue. Because v_j is a switch-vertex, h_j and h_{j+1} have different colors, which implies that h_{j+1} is red. Since h_j is blue and h_{j+1} is red, we infer from Lemma 3.7 that v_j is blue. Therefore, v_i and v_j have different colors, i.e., two consecutive switch-vertices along ∂f alternate in color, which implies that f is well-colored. □

Let f be a well-colored face of G_M^n and let P_f be the set of switch-vertices of f. A *switch-matching* M_f of f is a BR-matching on P_f such that every edge of M_f is contained in f (or on ∂f). Since f is well-colored, the sequence of switch-vertices along ∂f alternates in color. Moreover, since f is a convex simple polygon, we can obtain M_f by connecting consecutive switch-vertices along ∂f. That is, every face of G_M^n admits a switch-matching.

Recall that a vertex is a switch-vertex in exactly one face of G_M^n by Lemma 3.6. Therefore, as every face of G_M^n is well-colored by Lemma 3.8, we can obtain a BR-matching compatible with M by taking the union of the switch-matchings of every face in G_M^n. However, this BR-matching may have more crossings with ℓ than M, so we need to be careful when matching the switch-vertices of G_M^n.

Lemma 3.9. *Let f be a well-colored face of G_M^n that crosses ℓ. There exists a switch-matching M_f on the switch-vertices of f such that at most one edge of M_f intersects ℓ.*

Proof. Since f is a convex polygon, ℓ intersects ∂f in exactly two points u and d. Assume without loss of generality that u lies above d; see Figure 7 (a). Notice that u and d are visible points in G_M^n lying on the line ℓ. Because the color-invariant holds in G_M^n by Lemma 3.4, u and d are not color-visible. So, without loss of generality, assume that u is blue when viewed from d and hence that d is red when viewed from u. Walk counterclockwise from u and d along ∂f and let r_u and r_d, respectively, be the first switch-vertex reached along this walk. By Lemma 3.7, we know that r_u is blue whereas r_d is red.

Recall that we want to construct a switch-matching M_f of f. Let V_L and V_R be the sets of switch-vertices in f that lie to the left and right, respectively, of the supporting line of $r_d r_u$, directed from r_d to r_u. Let $\pi \in \{L, R\}$. Because $r_d r_u$ is a bichromatic edge, V_π contains an even number of switch-vertices, half of them red and half of them blue. As V_π is a set in convex position, there exists a BR-matching on V_π. Further, the convex hull of V_π does not intersect ℓ; see Figure 7 (b). Thus, for each BR-matching M_π on V_π no edge intersects ℓ.

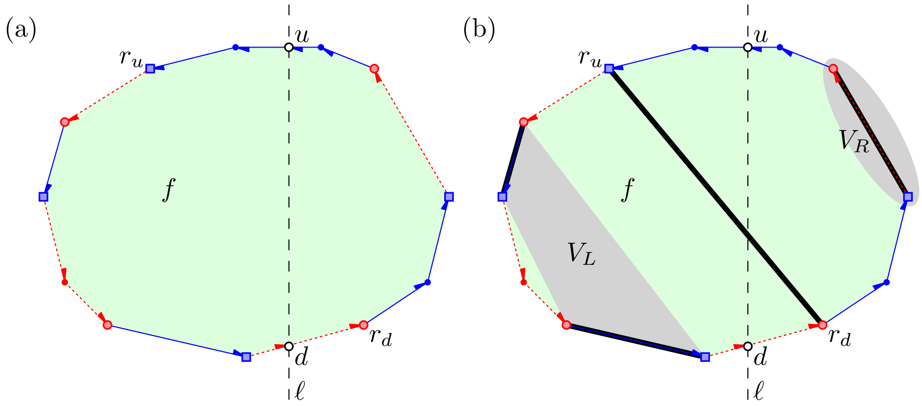

Figure 7: (a) A well-colored face f of G^n_M whose boundary intersects ℓ at points u and d. The vertices r_u and r_d are the first switch-vertices encountered when walking along the boundary of f counterclockwise from u and d, respectively. (b) The sets V_L and V_R contain the switch-vertices of f lying to the left and right, respectively, of $r_d r_u$. Moreover, the convex hulls of V_L and V_R are contained to the left and right, respectively, of ℓ. (V_R contains only two points and thus its convex hull has no area.) The bold edges exemplify one switch-matching M_f.

We obtain a switch-matching M_f of f by taking the union of the edges of M_L and M_R, and adding the edge $r_d r_u$, which is the only edge in M_f intersecting ℓ. □

3.4 Putting things together

We proceed by showing how to obtain a BR-matching M' on P such that M' and G^n_M are compatible (and hence, M' and M are compatible) and M' has fewer edges intersecting ℓ than M has. Recall that $C_{M,\ell}$ is the sequence of edges of M that intersect ℓ.

Lemma 3.10. *Let M be a BR-matching on P and let ℓ be a ham-sandwich cut of P. There exists a BR-matching M' compatible with M such that $|C_{M',\ell}| \leq |C_{M,\ell}| - 2$.*

Proof. For each face f of G^n_M, consider a switch-matching M_f on the switch-vertices of f, such that the edges of M_f have the minimum number of intersections with ℓ. Let M' be the BR-matching which is the union of the edges of all these switch-matchings M_f for all faces f of G^n_M. Because every switch-matching M_f is contained in its respective face f, M' and G^n_M are compatible. Moreover, since M is contained in the boundary of G^n_M, M' and M are compatible.

Observe that edges of M_f can intersect ℓ only if f crosses ℓ. By Lemma 3.5, there are at most $k-1$ faces of G^n_M that cross ℓ, where $k = |C_{M,\ell}|$. Furthermore, by Lemma 3.9, each of these faces admits a switch-matching having at most one edge intersecting ℓ. Therefore, M' contains at most $k-1$ edges that intersect ℓ. However, every BR-matching must have an even number of edges that intersect ℓ [4]. Therefore, M' contains at most $k-2$ edges that intersect ℓ, proving our result. □

We are now ready to provide the proof of Lemma 2.1 which is restated below.

Lemma 2.1. *Let $P = B \cup R$ be a bichromatic set of $2n$ points in the plane in general position such that $|B| = |R| = n$. For every BR-matching M and every ham-sandwich cut ℓ of P, there exists a BR-matching M^ℓ such that M and M^ℓ are $\lfloor n/2 \rfloor$-compatible and no edge of M^ℓ intersects ℓ.*

Proof. Let $M_0 = M$ and $k = |C_{M,\ell}|$. We know from Lemma 3.10 that for each BR-matching M_i with $|C_{M_i,\ell}| > 0$ there exists a BR-matching M_{i+1}, such that M_i and M_{i+1} are compatible and $|C_{M_{i+1},\ell}| \leq |C_{M_i,\ell}| - 2$. Hence, there exists a transformation $M = M_0, \ldots, M_t = M^\ell$ of length t between M and M^ℓ, where M^ℓ contains no edge intersecting ℓ. As $|C_{M_{i+1},\ell}| \leq |C_{M_i,\ell}| - 2$, for $0 \leq i \leq t-1$, we conclude that $t \leq k/2 \leq n/2$, i.e., M and M^ℓ are $\lfloor n/2 \rfloor$-compatible. □

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