k^{th}-order Voronoi Diagrams

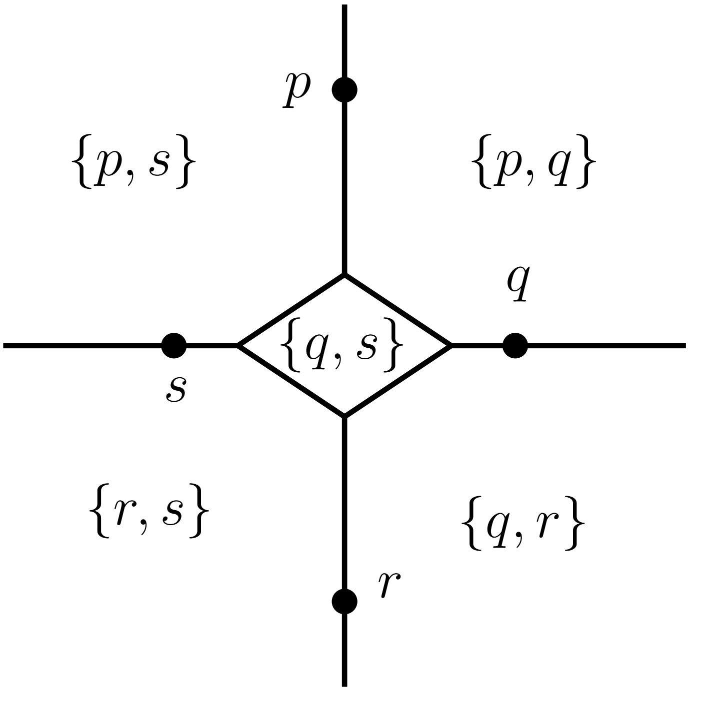
References:

- D.-T. Lee, "On k-nearest neighbor Voronoi Diagrams in the plane," *IEEE Transactions on Computers*, Vol. 31, No. 6, pp. 478–487, 1982.
- B. Chazelle and H. Edelsbrunner, "An improved algorithm for constructing kth-order Voronoi Diagram," *IEEE Transactions on Computers*, Vol. 36, No.11, pp. 1349–1454, 1987.
- C. Bohler, P. Cheilaris, R. Klein, C.-H. Liu, E. Papadopoulou, and M. Zavershynskyi, "On the complexity of higher order abstract Voronoi diagrams," Proceedings of the 40th International Colloquium on Automata, Languages and Programming (ICALP'13), pp. 208–219, 2013.

Given a set S of n point sites in the Euclidean plane, the k^{th}-order Voronoi diagram $\boldsymbol{V_k(S)}$ is a planar subdivision such that

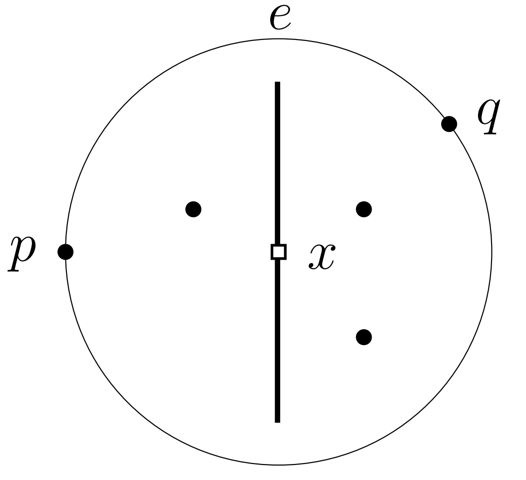
- each region is associated with a k-element subset H of S and denoted by $\mathrm{VR}_k(H, S)$.
- all points in $\mathrm{VR}_k(H, S)$ share the same k nearest sites H among S.

$\boldsymbol{V_2(S)}$



Property 1

Consider a Voronoi edge e between $\mathrm{VR}_k(H_1, S)$ and $\mathrm{VR}_k(H_2, S)$.
H_1 and H_2 only differ by one site.
Let $H_1 \setminus H_2$ be $\{p\}$ and $H_2 \setminus H_1$ be $\{q\}$.
For all points $x \in e$, $H_1 \cap H_2$ are the $k-1$ nearest sites of x and both p and q are the k^{th} nearest sites of x.



$k = 4$
$|H_1 \cap H_2| = 3$

General Position Assumption

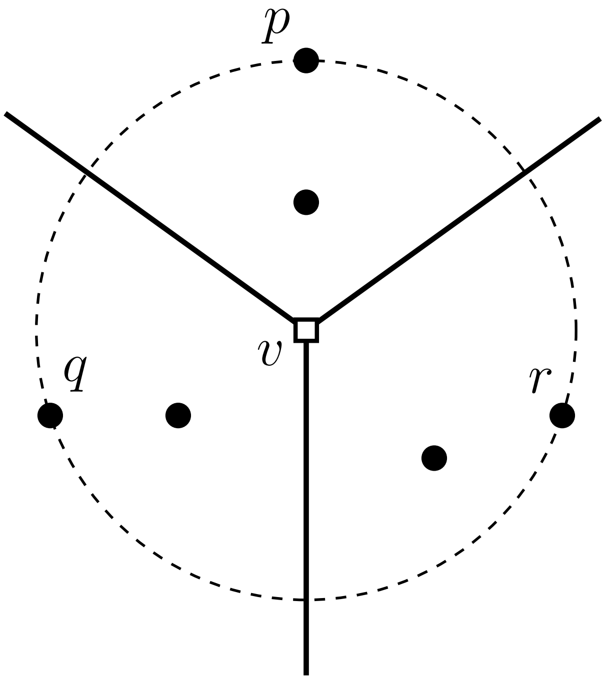
- no more than than sites are on the same line
  $\rightarrow V_k(S)$ is connected.
- no more than three sites are on the same circle
  $\rightarrow$ the degree of a Voronoi vertex is exactly 3.

Definition 1

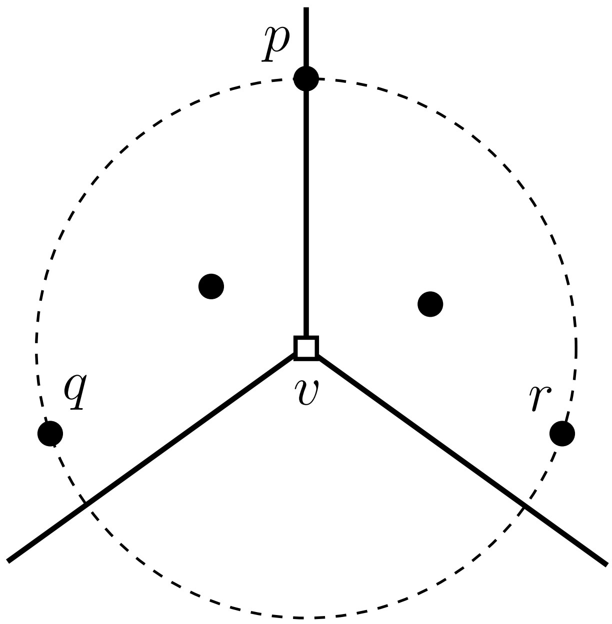
Consider a Voronoi vertex v among $\mathrm{VR}_k(H_1, S)$, $\mathrm{VR}_k(H_2, S)$, and $\mathrm{VR}_k(H_3, S)$.

- v is **new** if $|H_1 \cup H_2 \cup H_3| = k + 2$. $H_1 = H \cup \{p\}$, $H_2 = H \cup \{q\}$, $H_3 = H \cup \{r\}$, where $|H| = k - 1$.
  $\rightarrow$ the circle centered at v and touching p, q, and r will exactly enclose the $k-1$ sites of H.
- v is **old** if $|H_1 \cup H_2 \cup H_3| = k+1$. $H_1 = H \cup \{p, q\}$, $H_2 = H \cup \{q, r\}$, $H_3 = H \cup \{p, r\}$, where $|H| = k - 2$.
  $\rightarrow$ the circle centered at v and touching p, q, and r will exactly enclose the $k-2$ sites of H.

Example



v is new
$k = 4$
$H_1 = H \cup \{p\}$
$H_2 = H \cup \{q\}$
$H_3 = H \cup \{r\}$
$|H| = 3$



v is old
$k = 4$
$H_1 = H \cup \{p, q\}$
$H_2 = H \cup \{q, r\}$
$H_3 = H \cup \{p, r\}$
$|H| = 2$

Proprety 2
v is a Voronoi vertex among $\mathrm{VR}_k(H_1, S)$, $\mathrm{VR}_k(H_2, S)$, and $\mathrm{VR}_k(H_3, S)$

(a) v is **new**
$\rightarrow$ v is an **old** Voronoi vertex among $\mathrm{VR}_k(H_1 \cup H_2, S)$, $\mathrm{VR}_k(H_2 \cup H_3, S)$, $\mathrm{VR}_k(H_3 \cup H_1, S)$.

(b) v is **old**
$\rightarrow$ v belongs to $\mathrm{VR}_k(H_1 \cup H_2 \cup H_3)$.

Property 3
Consider an edge e between $\mathrm{VR}_k(H_1, S)$ and $\mathrm{VR}_k(H_2, S)$.
Then all points $x \in e$ belong to $\mathrm{VR}_k(H_1 \cup H_2)$.
Sketch of proof:
Let $H_1 \setminus H_2$ be $\{p\}$ and $H_2 \setminus H_1$ be $\{q\}$. Since e is a part of the bisector $B(p, q)$ between p and q, the circle centered at x and touching p and q will enclose all the $k-1$ sites of $H_1 \cap H_2$. Therefore, $(H_1 \cap H_2) \cup \{p, q\} = H_1 \cup H_2$ are exactly the $k+1$ nearest sites of x.

Definition 2
For a Voronoi edge e of $V_k(S)$, if one endpoint of e is an old Voronoi vertex, e is called **old**; otherwise, e is called **new**.

Property 4
New vertices of $V_k(S)$ decompose $V_k(S)$ into two kinds of connected components:

1. a new Voronoi edge
2. a connected subgraph whose internal nodes are old Voronoi vertices

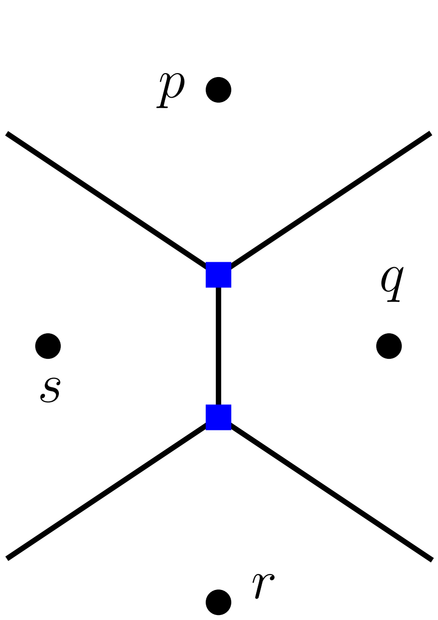
Each kind induces a Voronoi region of $V_{k+1}(S)$. (The former comes from Property 2 (a) and Property 3, and the latter comes from Property 2(b) and Property 3.)

Definition 3
For $i > 1$, Voronoi regions $\mathrm{VR}_i(H, S)$ of $V_i(S)$ can be categorized into two types:

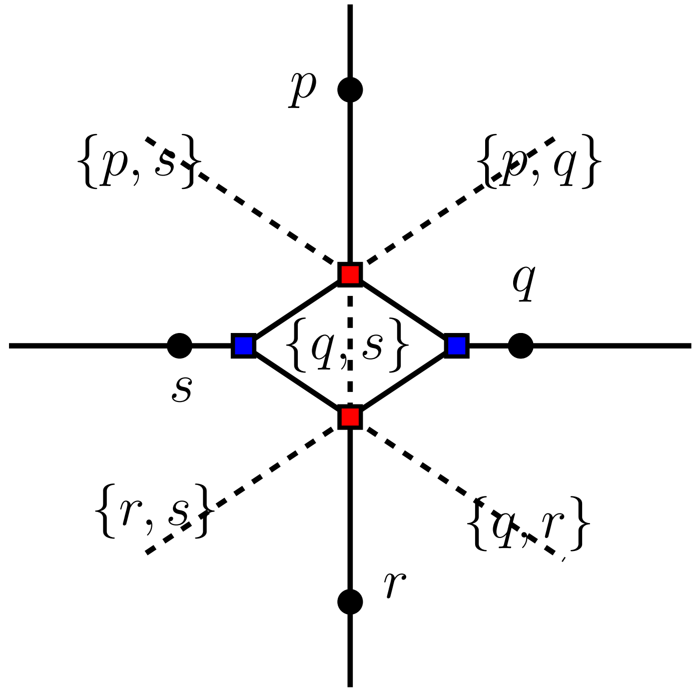
- **type-1**: $\mathrm{VR}_i(H, S)$ contains one new edge of $V_{i-1}(S)$.
- **type-2**: $\mathrm{VR}_i(H, S)$ contains old vertices of $V_{i-1}(S)$.

Example

Type-1



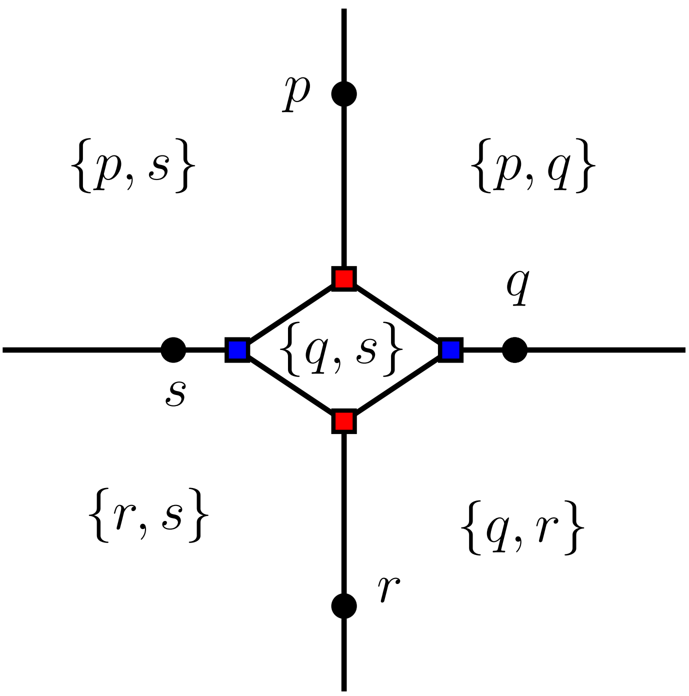
$V_1(S)$



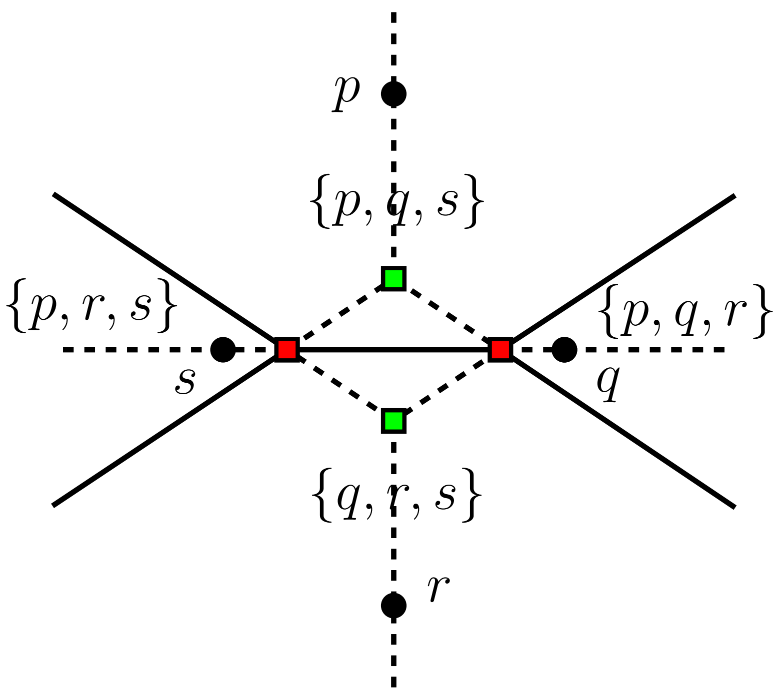
$V_2(S)$

$\mathrm{VR}_2(\{q,s\},S)$ is a type-1 region because it contains one new edge of $V_1(S)$

Type-2



$V_2(S)$



$V_3(S)$

Both $\mathrm{VR}_3(\{p,q,s\},S)$ and $\mathrm{VR}_3(\{q,r,s\},S)$ are type-2 regions because they contain old vertices of $V_2(S)$

Lemma 1
For $i > 1$, $V_{i-1}(S) \cap \mathrm{VR}_i(H,S)$ is a tree. $V_{i-1}(S) \cap \mathrm{VR}_i(H,S)$ is $V_{i-1}(H) \cap \mathrm{VR}_i(H,S)$
Sketch of proof

- all points in $\mathrm{VR}_i(H,S)$ share the same i nearest sites.
- $V_{i-1}(S)$ partitions $\mathrm{VR}_i(H,S)$ into at most t sub-regions, and $t < i$.
- For $1 \le j \le t$, let R_j be a sub-region of $V_{i-1}(S) \cap \mathrm{VR}_i(H,S)$, let H_j be the $(i-1)$-element subset of S such that $R_j = \mathrm{VR}_{i-1}(H_j,S) \cap \mathrm{VR}_i(H,S)$, and let $H \setminus H_j$ be $\{s_j\}$.
- For all points x in R_j, H_j are the $(i-1)$ nearest sites of x, and s_j is the i^{th} nearest site of x.
- In other wods, s_j is the farthest site of x among H.
- $V_{i-1}(S)$ forms the fartheset site Voronoi diagram of H inside $\mathrm{VR}_i(H,S)$, i.e., $V_{i-1}(S) \cap \mathrm{VR}_i(H,S) = V_{i-1}(H) \cap \mathrm{VR}_i(H,S)$.
- The farthest-site Voronoi diagram is a tree
- By Property 4, $V_{i-1}(S) \cap \mathrm{VR}_i(H,S)$ is a connected component, and thus $V_{i-1}(H) \cap \mathrm{VR}_i(H,S)$ is a tree.

Corollary 1
If $\mathrm{VR}_i(H,S)$ contains m old Voronoi vertices of $V_{i-1}(S)$, $\mathrm{VR}_i(H,S)$ contains $2m+1$ old Voronoi edges of $V_{i-1}(S)$.
Sketch of proof

- By the generation position assumption, the degree of a Voronoi vertex is 3.
- By Lemma 1, $V_{i-1}(S) \cap \mathrm{VR}_i(H,S)$ is a tree.

Euler formular for a planar subdivision

$$v - e + f = 1 + c,$$

where v is # of vertices, e is # of edges, f is # of faces, and c is # of connected component

Corollary 2
Under the general position assumption,

- $E_k = 3(N_k - 1) - \mathcal{S}_k$
- and $I_k = 2(N_k - 1) - \mathcal{S}_k$,

where E_k is # of edges, I_k is # of vertices, N_k is # of faces, and $\mathcal{S}_k$ is # of unbounded faces of $V_k(S)$.

Theorem 1
Given a set S of n point sites in the Euclidean plane, the total number N_k of regions in $V_k(S)$ is $2k(n-k)+k^2-n+1-\sum_{i=1}^{k-1}\mathcal{S}_i$, where $\mathcal{S}_i$ is # of unbounded regions in $V_i(S)$, and $\mathcal{S}_0$ is defined to be 0.

proof

- I_i, I_i' and I_i'' are # of vertices, new vertices, and old vertices of $V_i(S)$, respectively.
- E_i, E_i' and E_i'' are # of edges, new edges, and old edges of $V_i(S)$, respectively.
- N_i, N_i' and N_i'' are # of regions, type-1 regions, and type-2 regions of $V_i(S)$, respectively.
- Since an old vertex of $V_{i+1}(S)$ is a new vertex of $V_i(S)$,

$$I_{i+1} = I_{i+1}' + I_{i+1}'' = I_{i+1}' + I_i'$$

$$\rightarrow I_{i+1}' = I_{i+1} - I_i'$$

- $I_1 = I_1'$, $E_1 = E_1'$, and $E_{i+1} = E_{i+1}' + E_{i+1}''$
- Order N_{i+2}'' type-2 regions of $V_{i+2}(S)$, let m_j be the number of old vertices of $V_{i+1}(S)$ inside the j^{th} type-2 region of $V_{i+2}(S)$, and let e_j be the number of edges of $V_{i+1}(S)$ inside the j^{th} type-2 region of $V_{i+2}(S)$.
- $\sum_{j=1}^{N_{i+2}''} m_j = I_{i+1}'' = I_i'$ and $\sum_{j=1}^{N_{i+2}''} e_j = E_{i+1}''$
- By Corollay 1,

$$E_{i+1}'' = \sum_{j=1}^{N_{i+2}''} e_j = \sum_{j=1}^{N_{i+2}''}(2m_j+1) = 2I_i' + N_{i+2}'' \rightarrow N_{i+2}'' = E_{i+1}'' - 2I_i'$$

- $$N_{i+2} = N'_{i+2} + N''_{i+2} = E'_{i+1} + (E''_{i+1} - 2I'_i) = E_{i+1} - 2I'_i$$
- $N_1 = n$ and $N_2 = E'_1 = E_1 = 3(n-1) - \mathcal{S}_1$.
- since $N_{i+2} = E_{i+1} - 2I'_i$, $E_i = 3(N_i - 1) - \mathcal{S}_i$, and $I_i = 2(N_i - 1) - \mathcal{S}_i$,
$$N_{k+2} = E_{k+1} - 2I'_k = 3(N_{k+1} - 1) - \mathcal{S}_{k+1} - 2I'_k$$
$$= 3(N_{k+1} - 1) - \mathcal{S}_{k+1} - 2\sum_{i=1}^{k}(-1)^{k-i} I_i$$
$$= 3(N_{k+1} - 1) - \mathcal{S}_{k+1} - 2\sum_{i=1}^{k}(-1)^{k-i}(2(N_i - 1) - \mathcal{S}_i)$$
- By induction on k,
$$N_k = 2k(n-k) + k^2 - n + 1 - \sum_{i=1}^{k-1} \mathcal{S}_i$$

Theorem 2
$N_k = O(k(n-k))$

- If $k \leq n/2$, by Theorem 1, N_k is trivially $O(k(n-k))$.
- If $k > n/2$, N_k depends on $\sum_{i=1}^{k-1} \mathcal{S}_i$
- Since $\sum_{i=1}^{n-1} \mathcal{S}_i = n(n-1)$, $\sum_{i=1}^{k-1} \mathcal{S}_i = n(n-1) - \sum_{i=k}^{n-1} \mathcal{S}_i$
- Since $\mathcal{S}_i = \mathcal{S}_{n-i}$, $\sum_{i=k}^{n-1} \mathcal{S}_i = \sum_{i=1}^{n-k} \mathcal{S}_i$
- $N_k = 2k(n-k) + k^2 - n + 1 - \sum_{i=1}^{k-1} \mathcal{S}_i$
$= 2k(n-k) + k^2 - n + 1 - n(n-1) + \sum_{i=k}^{n-1} \mathcal{S}_i$
$= N_k = 2k(n-k) + k^2 - n + 1 - n(n-1) + \sum_{i=1}^{n-k} \mathcal{S}_i$
- Since $\sum_{i=1}^{n-k} \mathcal{S}_i \leq (n-k)n$ (recal # of $\leq k$-set),
$$N_k \leq 2k(n-k) + k^2 - n + 1 - n(n-1) + (n-k)n = k(n-k) + 1$$

Iterative Construction

Theorem 3
$V_{i+1}(S)$ can be obtained from $V_i(S)$ by taking $\mathrm{VR}_i(H, S) \cap V_1(S \setminus H)$ for all $H \subseteq S$ such that $V_i(H, S)$ is non-empty.

Sketch of proof

- $V_1(S \setminus H) \cap \mathrm{VR}_i(H, S) = V_{i+1}(S) \cap \mathrm{VR}_i(H, S)$
  - all points in $\mathrm{VR}_i(H, S)$ share the same i nearest sites H among S
  - all points in $\mathrm{VR}_1(p, S \setminus H)$ share the same nearest site p among $S \setminus H$.
  - all points in $\mathrm{VR}_1(p, S \setminus H) \cap \mathrm{VR}_i(H, S)$ share the same i nearest sites H and $(i+1)^{\mathrm{st}}$ nearest site p among S, implying that $\mathrm{VR}_1(p, S \setminus H) \cap \mathrm{VR}_i(H, S) \subseteq \mathrm{VR}_{i+1}(H \cup \{p\}, S)$
  - It is trivial that $\mathrm{VR}_{i+1}(H \cup \{p\}, S) \cap \mathrm{VR}_i(H, S) \subseteq \mathrm{VR}_1(p, S \setminus H)$,
  - $\mathrm{VR}_1(p, S \setminus H) \cap \mathrm{VR}_i(H, S) = \mathrm{VR}_{i+1}(H \cup \{p\}, S) \cap \mathrm{VR}_i(H, S)$ for $\forall p \in H$

Corollary 3
Assume $\mathrm{VR}_i(H, S)$ has m adjacent regions $\mathrm{VR}_i(H_j, S)$, $1 \leq j \leq m$. Let Q be $\bigcup_{1 \leq j \leq m} H_j \setminus H$. Then $V_{i+1}(S) \cap \mathrm{VR}_i(H, S) = V_1(Q) \cap \mathrm{VR}_i(H, S)$
The proof will be an exercise.

Compute $V_{i+1}(S)$ from $V_i(S)$

- For each nonempty region $\mathrm{VR}_i(H, S)$, compute $V_1(Q) \cap \mathrm{VR}_i(H, S)$ where $\mathrm{VR}_i(H, S)$ has m adjacent regions $\mathrm{VR}_i(H_j, S)$, $1 \leq j \leq m$, and Q is $\bigcup_{1 \leq j \leq m} H_j \setminus H$.

Lemma 2
$V_{i+1}(S)$ can be obtained from $V_i(S)$ in $O(i(n-i)\log n)$ time.
Sketch of proof

- $V_1(Q)$ can be computed in $|Q|\log|Q|$ time.
- $|Q| \leq |\partial \mathrm{VR}_i(H,S)|$ where $\partial \mathrm{VR}_i(H,S)$ is the boundary of $\mathrm{VR}_i(H,S)$
- $$\sum_{H\subset S,|H|=i,\mathrm{VR}_i(H,S)\neq\emptyset} O(|\partial \mathrm{VR}_i(H,S)|\log|\partial \mathrm{VR}_i(H,S)|)$$
$$=\log n \sum_{H\subset S,|H|=i,\mathrm{VR}_i(H,S)\neq\emptyset} O(|\partial \mathrm{VR}_i(H,S)|)$$
$$=O(i(n-i)\log n)$$

Theorem 4
$V_k(S)$ can be computed in $O(k^2 n\log n)$ time.
Sketch of proof

- $V_1(S)$ can be computed in $O(n\log n)$
- $O(n\log n)+\sum_{i=1}^{k-1} O(i(n-i)\log i) = O(k^2 n\log n)$.

Construction by Geometric Duality and Arrangement

Definition 4 (Bisectors)

- For two sites, $p, q \in S$, the bisector $B(p, q)$ is $\{x \in \mathtt{R}^2 \mid d(x, p) = d(x, q)\}$.
- For a site $p \in S$, let B_p be $\{B(p, q) \mid q \in S \setminus \{p\}\}$.

Definition 5
For a site $p \in S$, the k-neighborhood of p is $\bigcup_{p\in H, H\subset S, |H|=k} \mathrm{VR}_k(H, S)$ and denoted by $\mathrm{VN}_k(p, S)$. $\mathrm{VN}_k(p, S)$.

Property 5

$$V_k(S) = \bigcup_{p\in S} \partial\mathrm{VN}_k(p, S)$$

Lemma 3
$\mathrm{VN}_k(p, S)$ is connected and each edge of $\partial\mathrm{VN}_k(p, S)$ is a part of the bisector $B(p, q)$ for some $q \in S \setminus \{p\}$.
The proof could be a bonus task.

Lemma 4
Consider an edge of $\partial\mathrm{VN}_k(p, S)$. For any point $x \in e$, $\overline{px}$ intersects exactly $k - 1$ bisectors of B_p.
Sketch of proof

- W.l.o.g, let e belong to $\mathrm{VR}_k(H_1, S) \cap \mathrm{VR}_k(H_2, S)$ and let p belong to $H_1 \setminus H_2$.
- It is clear that $H_1 \setminus \{p\}$ are the $k - 1$ nearest sites of x.
- For any $q \in H_1 \setminus \{p\}$, x belongs to $D(q, p)$, i.e., $\overline{px}$ intersects $B(p, q)$. For any $q \in S \setminus H_1$, x does not belongs to $D(q, p)$, i.e., $\overline{px}$ does not intersects $B(p, q)$.

Definition 6

- Given a set L of lines in the plane, let $A(L)$ be the arrangement fromd by L.
- For a point x in a face of $A(L)$, an edge e of $A(L)$ is at level i from x if for any point $y \in e$, $\overline{yx}$ intersects exactly $i-1$ lines of L.
- The i-skeleton $\mathrm{SK}_i(x, L)$ is the collection of edges in $A(L)$ whose level from x is i.

Lemma 5

$$\partial \mathrm{VN}_k(p, S) = \mathrm{SK}_k(p, B_p)$$

Therefore, computing $V_k(S)$ is equivalent to computing $\mathrm{SK}_k(p, B_p)$ for all sites $p \in S$.
Hereafter, we translate S such that p is located at $(0, 0)$, and let L be B_p.
If we know all the vertices of $\mathrm{SK}_k(p, L)$ and their order along $\mathrm{SK}_k(p, L)$ (clockwise or counterclockwise, we can compute $\mathrm{SK}_k(p, L)$

Lemma 6 Under the general position assumption, for a vertex v of $\mathrm{SK}_k(p, B_p)$, $\overline{pv}$ intersects $k-1$ or $k-2$ lines of B_p.

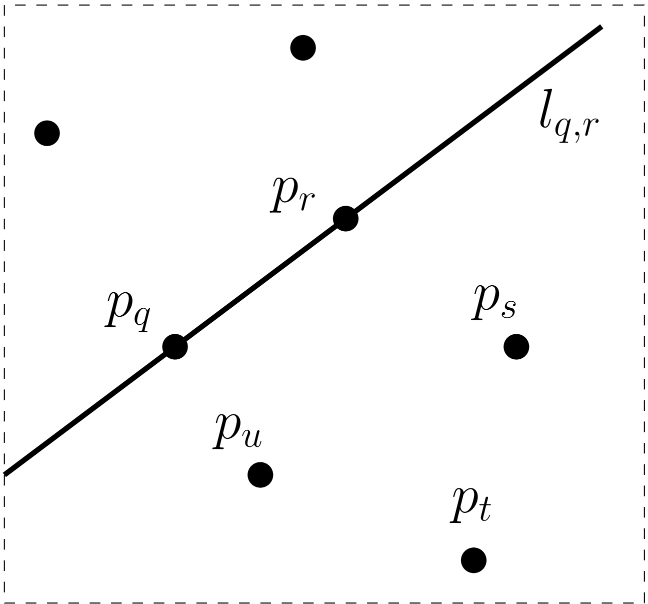
Geometric Duality
Consider a function Ψ. For a point $x = (a, b)$ except the origin, $\Psi(x)$ is a line $: ax_1 + bx_2 = 1$, and for a line $l : ax_1 + bx_2 = 1$, $\Psi(x)$ is a point (a, b).

Lemma 7

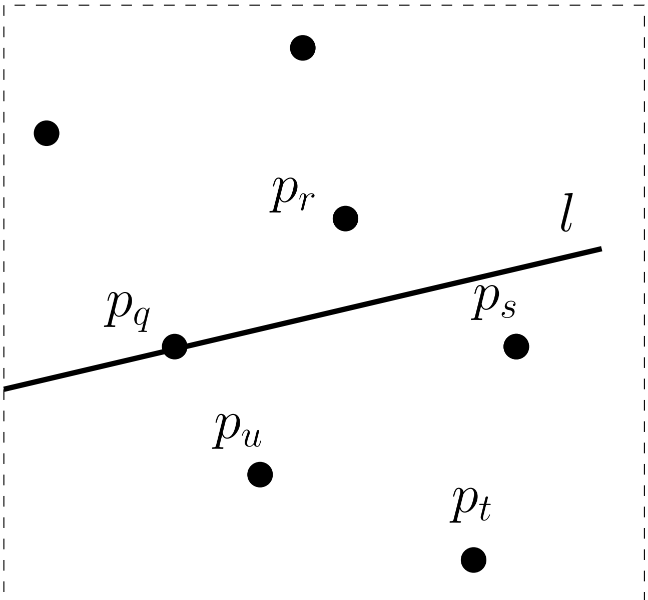
- For an edge e of $\mathrm{SK}_k(p, B_p)$ and any point $x \in e$, $\Psi(x)$ partitions the plane such that one half-plane contains the origin and exactly $k-1$ points of $\Psi(B_p)$.
- For a vertex v of $\mathrm{SK}_k(p, B_p)$, $\Psi(v)$ partitions the plane such that one half-plane contains the origin and $k-1$ or $k-2$ points of $\Psi(B_p)$.

Example

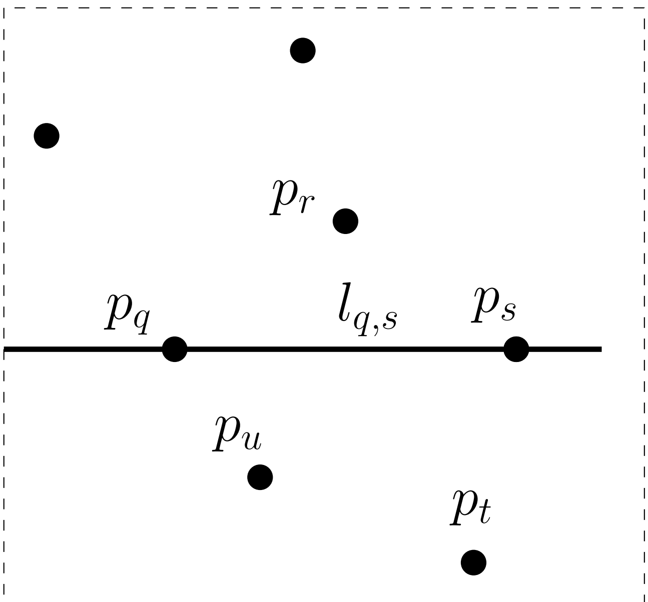
For $q \in S \setminus \{p\}$, let p_q be $\Psi(B(p,q))$. Consider $n = 8$ and $k = 4$.



$l_{q,r}$ corresponds to a new Voronoi vertex among $\mathrm{VR}_k(H_1, S)$, $\mathrm{VR}_k(H_2, S)$, and $\mathrm{VR}_k(H_3, S)$, where $H_1 = H \cup \{p\}$, $H_2 = H \cup \{q\}$, $H_3 = H \cup \{r\}$, and $H = \{s, t, u\}$.



l corresponds to a point on a Voronoi edge between $\mathrm{VR}_k(H_1, S)$and $\mathrm{VR}_k(H_2, S)$, where $H_1 = H \cup \{p\}$, $H_2 = H \cup \{q\}$, and $H = \{s, t, u\}$.

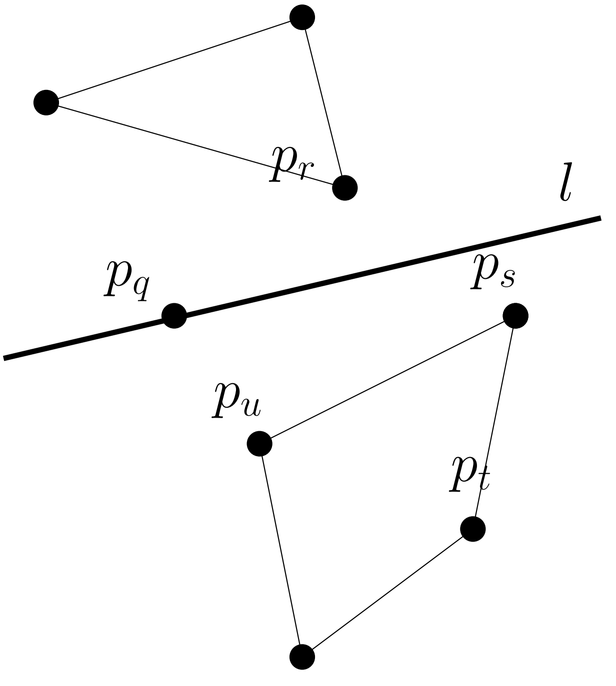


$l_{q,s}$ corresponds to an old Voronoi vertex among $\mathrm{VR}_k(H'_1, S)$, $\mathrm{VR}_k(H'_2, S)$, and $\mathrm{VR}_k(H'_3, S)$, where $H'_1 = H' \cup \{p, s\}$, $H'_2 = H' \cup \{q, s\}$, $H'_3 = H \cup \{p, q\}$, and $H' = \{t, u\}$. (Note $H'_1 = H_1$ and $H'_2 = H_2$.)

Let v_1, v_2, ... be a sequence of vertices of $\mathrm{SK}_k(p, B_p)$ along the counterclockwise order.
We consider how to compute v_{i+1} from v_i.

- W.l.o.g., we let v_i be the intersection between $B(p, q)$ and $B(p, r)$ and v_{i+1} be $B(p, q)$ and $B(p, s)$. But we do not know s.
- Similarly, for each $q \in S \setminus \{p\}$, let p_q be $\Psi(B(q, p))$.
- $\Psi(v_i)$ is a straight line passing through p_q and p_r.
- Let l be $\Psi(v_i)$, and rotate l at p_q in the direction such that one half-plane contains the origin and exactly $k-1$ points of $\Psi(B_p)$.
- The rotation will hit p_s first and we obtain v_{i+1}.
- During the rotation, l partition $\Psi(B_p \setminus \{B(p, q)\})$ into the same 2 sets.

Property 6
Let e be an edge of $\mathrm{SK}_k(p, S)$ and belong to $B(p, q)$. Let v be an endpoint of e and v be an intersection between $B(p, q)$ and $B(p, s)$. For any point $x \in e$, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the 2-partition of $\Psi(B_p \setminus \{B(p, q)\})$ formed by $\Psi(x)$. Then, $\Psi(B(p, s))$ must be one of four tangent points between $\Psi(B(p, q))$ and the two convex hulls of $\mathcal{P}_1$ and $\mathcal{P}_2$.



Lemma 8
$\mathrm{SK}_k(p, B_p)$ can be constructed in $O(n \log n + |SK_k(p, B_p)| \log n)$ time.
Sketch of proof

- After the sorting, it takes $O(n)$ time to compute a vertex of $\mathrm{SK}_k(p, B_p)$ and then view the vertex as the begining vertex v_1.
- It sufficient to analyze the time for computing v_{i+1} from v_i.
- Assume that v_i is an intersection between $B(p, q)$ and $B(p, r)$.
- Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the 2-partion of $\Psi(B_p \setminus \{p\})$ formed by $\Psi(v_i)$ and let $\mathcal{P}_1$ belong to the half-plane containing the origin.
- If v_i is a new Voronoi vertex, $|\mathcal{P}_1| = k - 1$.
  - let l be $\Psi(v_i)$
  - rotate l at $\Psi(B(p, q))$ such that $\mathcal{P}_1$ and $\Psi(B(p, r))$ belongs to different half-planes formed by l.
  - Determine that l first touches the convex hull of $\mathcal{P}_1$ or that of $\mathcal{P}_2 \cup \{\Psi(B(p, r))\}$
  - Let $\Psi(B(p, s))$ be the first touched point of the first touched convex hull. Then v_{i+1} is the intersection between $B(p, q)$ and $B(p, s)$.
- Otherwise, v_i is an old Voronoi vertex, and $|\mathcal{P}_1| = k - 2$.
  - let l be $\Psi(v_i)$
  - rotate l at $\Psi(B(p, q))$ such that $\mathcal{P}_1$ and $\Psi(B(p, r))$ belong to the same half-plane formed by l.
  - Determine that l first touches the convex hull of $\mathcal{P}_1 \cup \{\Psi(B(p, r))\}$ or that of $\mathcal{P}_2$
  - Let $\Psi(B(p, s))$ be the first touched point of the first touched convex hull. Then v_{i+1} is the intersection between $B(p, q)$ and $B(p, s)$.
- Brodal and Jacob proposed a dynamic structure for the convex hulls allowing insertion, deletion, and tangent query in amorted $O(\log n)$ time.
- It takes $O(n \log n)$ time to compute the two initial convex hulls.
- There are $O(|SK_k(p, B_p))|$ insertions, deletions, and tangent queries.

Theorem 5
$V_k(S)$ can be computed in $O(n^2 \log n + k(n-k) \log n)$ time.
sketch of proof

- $V_k(S) = \bigcup_{p \in S} SK_k(p, B_p)$.
- $\sum_{p \in S} O(n \log n + |SK_k(p, B_p)| \log n) = O(n^2 \log n + k(n-k) \log n)$