Thomas Kesselheim
Matthias Buttkus

3rd June 2019
Due date: In the lecture on 19th June 2019, 12:15

Algorithmic Game Theory and the Internet

Summer Term 2019

Exercise Set 10

Exercise 1: (3+3 Points)
Consider the following single-item auctions with two bidders whose valuations are drawn independently from a uniform distribution on the interval $[0, 1]$.

(a) Show that the expected revenue of a second-price auction is $\frac{1}{3}$.

(b) Now, define a second-price auction with *reserve price* p. Let v_1 and v_2 be the valuations of the bidders. The allocation and payment rule will be determined according to the following cases:

1. $\min\{v_1, v_2\} \geq p$: Like in the second price auction.
2. $\max\{v_1, v_2\} < p$: Nobody gets the item and no payments.
3. $v_1 \geq p > v_2$: Bidder 1 gets the item and has to pay p.
4. $v_2 \geq p > v_1$: Analogous to 3.

Show that using a reserve price of $\frac{1}{2}$ the second-price auction generates an expected revenue of $\frac{5}{12}$.

Do not make use of the results of Lecture 17 in order to solve subtasks (a) and (b).

Hint: For each $y > 0$ calculate the probability of the event that the revenue is at least y. Afterwards, make use of it in order to calculate the expected revenue.

Exercise 2: (1+3 Points)
Once again, consider a single-item auction with two bidders whose valuations are drawn independently from a uniform distribution over $[0, 1]$.

(a) Prove that the random variables $\varphi_i(v_i)$ are distributed according to a uniform distribution on $[-1, 1]$.

(b) Utilize subtask (a) and the results of the lecture in order to determine the expected revenue of a second-price auction with reserve price $p \in [0, 1]$.

Exercises 3 and 4 on the next page.

Exercise 3: (2+2+2 Points)
Determine the virtual value function φ of the following probability distributions.

(a) Uniform distribution on the interval $[a, b]$.

(b) Exponential distribution with rate $\lambda > 0$ (defined on $[0, \infty)$).

(c) The distribution given by the cumulative distribution function $F(v) = 1 - \frac{1}{(v+1)^c}$ defined on the interval $[0, \infty)$, where $c > 0$ is considered to be an arbitrary constant.

Which of the stated distributions are regular?

Exercise 4: (4 Points)
State an example such that the allocation function which maximizes the virtual welfare is *not truthful*. For this purpose, state a distribution together with a pair v_i, b_i such that $u_i((v_i, b_{-i}), v_i) < u_i((b_i, b_{-i}), v_i)$.

Hint: It suffices to consider a single bidder.