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Algorithms and Uncertainty

Summer Term 2020

Exercise Set 1

If you want to hand in your solutions for this problem set, please send them via email to alexander.braun@uni-bonn.de *- make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.*

Exercise 1: (2+2+2 Points)

We want to recall the basics of linear programming. In the lecture, we have seen the Set Cover problem of which we now consider a special case: the *Vertex Cover* problem. The universe consists of edges in a graph which can be covered by their incident vertices. More formally, a vertex cover is a set of vertices $S \subseteq V$ such that for all $e = \{u, v\} \in E$ either $u \in S$ or $v \in S$. We are interested in finding a vertex cover of minimum size. For now, we restrict to bipartite graphs, i.e. graphs $G = (V, E)$ with $V = A \cup B$.

(a) Give the integer program of the Vertex Cover problem and its LP relaxation.

(b) Give the dual program to the LP from (a).

Additionally, consider the maximum matching problem in the bipartite version, i.e. given a bipartite graph $G = (V, E)$ with $V = A \cup B$ our goal is to compute a maximum matching where a matching is a set of edges $M \subseteq E$ such that no two edges in M share a common vertex.

(c) Compare the dual program from (b) to the LP relaxation of the maximum matching problem for bipartite graphs. Do you notice any similarities?

Exercise 2: (2+2+2+2 Points)

Consider the following Set Cover instance: $U = \{1, 2, 3\}$ and $\mathcal{S} = \{A, B, C\}$ with $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$, $c_A = c_B = 3$, $c_C = 4$.

(a) Give an optimal integral solution.

(b) Give a fractional primal solution of cost at most 5.

(c) Give a dual solution of value at least 5.

(d) Use your solution of (c) to show optimality of your solution of (b). To this end, sum up the primal constraints in a suitable way. (Your solution should be in the spirit of proof of weak duality but not use the statement of the lemma itself.)

Exercise 3: (3+4 Points)

Given an instance of Set Cover, let $f = \max_{e \in U} |\{S \in \mathcal{S} \mid e \in S\}|$ denote the *frequency* of the set system.

(a) Consider the unweighted version of Online Set Cover, i.e., $c_S = 1$ for all $S \in \mathcal{S}$, and the following algorithm: Upon arrival of element e, if $\sum_{S:e \in S} x_S = 0$, set $x_S = 1$ for all S with $e \in S$ and $y_e = 1$. Otherwise set $y_e = 0$. Show that this algorithm is f-competitive by using Lemma 2.7.

(b) Now, we generalize the algorithm from (a) to the weighted version. Let $g_e = \max\{0, 1 - \sum_{S:e \in S} x_S\}$ and let S_e be the cheapest set covering e. For each S that covers e, increase x_S by $\frac{c_{S_e}}{c_S} g_e$ and set $y_e = c_{S_e} g_e$. Show that this algorithm is f-competitive by using Lemma 2.7.