Online Set Cover: Fractional Algorithm

Thomas Kesselheim **Preview** Last Update: April 27, 2020

Last time, we introduced the online set cover problem and its fractional relaxation. Today, we will consider only the relaxed problem and devise online algorithms. So, our goal is to solve the following kind of linear program online.

$$\begin{aligned} \text{minimize} \sum_{S \in \mathcal{S}} c_S x_S & \\ \text{subject to} \sum_{S \colon e \in S} x_S \geq 1 & \qquad \text{for all } e \in U \\ x_S \geq 0 & \qquad \text{for all } S \in \mathcal{S} \end{aligned}$$

We have to maintain a feasible solution x to the linear inequalities. In the t-th step, the t-th element arrives and therefore we get to know the t-th constraint. Possibly, the solution $x^{(t-1)}$ we had so far is infeasible now. In this case, we may only *increase* variables to get to the solution $x^{(t)}$, which is feasible again.

Recall the dual of the set cover LP

$$\begin{aligned} \text{maximize} \sum_{e \in U} y_e & \\ \text{subject to} \sum_{e \in S} y_e \leq c_S & \qquad \text{for all } S \in \mathcal{S} \\ y_e \geq 0 & \qquad \text{for all } e \in U \end{aligned}$$

We will use a primal-dual algorithm. That is, besides maintaining a primal solution x, we will also maintain a dual solution y. In step t, variable y_t is added to the dual LP and we can only set its value. We will make use of the following lemma.

Lemma 3.1. *If*

(a) in every step t the primal increase is bounded by β times the dual increase, that is

$$P^{(t)} - P^{(t-1)} \leq \beta y_t \ , \text{ where } P^{(t)} = \sum_{S \in \mathcal{S}} c_S x_S^{(t)}$$

and,

(b) $\frac{1}{\gamma} y$ is dual feasible,

then the algorithm is $\beta\gamma$-competitive.

1 Fractional Ski Rental

To understand the design principle of primal-dual online algorithms, we will consider the fractional variant of Ski Rental first. Recall the LP relaxation

$$\begin{aligned} \text{minimize } B x_{\text{buy}} + \sum_t x_{\text{rent},t} & \\ \text{subject to } x_{\text{buy}} + x_{\text{rent},t} \geq 1 & \qquad \text{for all } t \\ x_{\text{buy}}, x_{\text{rent},t} \geq 0 & \qquad \text{for all } t \end{aligned}$$

and its dual

$$\begin{aligned} \text{maximize } & \sum_t y_t \\ \text{subject to } & \sum_t y_t \leq B \\ & y_t \leq 1 && \text{for all } t \\ & y_t \geq 0 && \text{for all } t \end{aligned}$$

Note that the optimal dual solution is always to set $y_1 = \ldots = y_B = 1$ and $y_{B+1} = \ldots = y_m = 0$. Our primal-dual algorithm will use exactly this dual solution. So, in Lemma 3.1, Property (b) holds with $\gamma = 1$. We now have to figure out how to update the primal solution so as to keep β as small as possible.

The dual does not increase in steps $t > B$. Therefore, to maintain Property (a), we may not increase the primal objective in these steps.

In step $t \leq B$, we increase x_{buy} some way. We have to set $x_{\text{rent},t}$ to fill the gap between x_{buy} and 1. So,

$$x_{\text{buy}}^{(t)} + x_{\text{rent},t}^{(t)} = 1 \ .$$

At the same time, the increase of the primal objective function has to be bounded by $\beta \cdot y_t = \beta$. That is

$$B(x_{\text{buy}}^{(t)} - x_{\text{buy}}^{(t-1)}) + x_{\text{rent},t}^{(t)} = \beta \ .$$

In combination, these two equalities give us

$$(B-1)x_{\text{buy}}^{(t)} - Bx_{\text{buy}}^{(t-1)} = \beta - 1 \ ,$$

or equivalently

$$x_{\text{buy}}^{(t)} = \frac{\beta - 1}{B-1} + \frac{B}{B-1} x_{\text{buy}}^{(t-1)} \ .$$

This recursion solves to

$$x_{\text{buy}}^{(t)} = \sum_{t'=0}^{t-1} \left(\frac{B}{B-1} \right)^{t'} \frac{\beta - 1}{B-1} = \frac{1 - \left(\frac{B}{B-1}\right)^t}{1 - \left(\frac{B}{B-1}\right)} \frac{\beta - 1}{B-1} = (\beta - 1) \left(\left(\frac{B}{B-1} \right)^t - 1 \right) \ .$$

Recall that $x_{\text{buy}}^{(B)} = 1$ because we may not increase primal variables after step B. So, we have to have

$$(\beta - 1) \left(\left(\frac{B}{B-1} \right)^B - 1 \right) = 1 \ ,$$

which is equivalent to

$$\beta = \left(1 - \left(1 - \frac{1}{B} \right)^B \right)^{-1} \ .$$

Note that $\left(1 - \left(1 - \frac{1}{B}\right)^B\right)^{-1} \leq \left(1 - \frac{1}{e}\right)^{-1} \approx 1.58$, so the algorithm is 1.58-competitive.

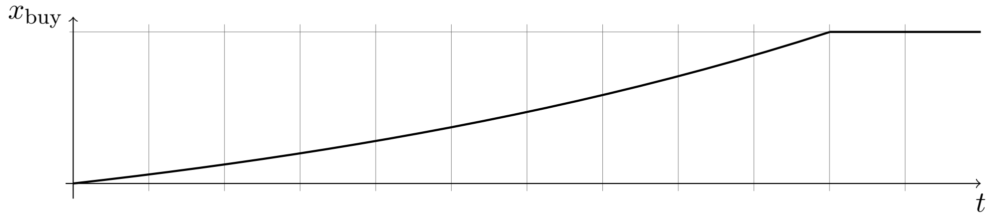

Figure 1: Increase of x_{buy} over time for $B = 10$.

2 Approach for Fractional Online Set Cover

Now, we turn to the more general Fractional Online Set Cover Problem. When choosing $x^{(t)}$ and y_t, our primary goal is that they have similar objective-function values so that Property (a) in Lemma 3.1 holds with a small β.

So, let us figure out what we would like to do. Suppose we are in step t. That is, element t arrives and we observe a new constraint $\sum_{S\colon t \in S} x_S \geq 1$ in the primal LP. In the dual, a new variable y_t arrives. Our current solution is $x^{(t-1)}$. It fulfills all constraints except maybe the new one. If also $\sum_{S\colon t \in S} x_S^{(t-1)} \geq 1$, there is nothing to do because we can keep the old solution as the new one by setting $x^{(t)} = x^{(t-1)}$, $y_t = 0$.

In the case $\sum_{S\colon t \in S} x_S^{(t-1)} < 1$, we will have to increase some variables to get a feasible $x^{(t)}$. Of course, $x^{(t)}$ will be more expensive than $x^{(t-1)}$. We reflect this additional cost in the value of y_t, all other dual variables remain unchanged.

Let us slowly increase x starting from $x^{(t-1)}$ and simultaneously y_t starting from 0. We do this in infinitesimal steps over continuous time.

We are at any point in time for which still $\sum_{S\colon t \in S} x_S < 1$. We increase x_S by dx_S. To account for the increased cost, we increase y_t by dy at the same time. The dual objective function increases by dy this way. This is at least $(\sum_{S\colon t \in S} x_S) dy$ because $\sum_{S\colon t \in S} x_S < 1$. Simultaneously, the primal objective function increases by $\sum_{S\colon t \in S} c_S dx_S$. If we set $dx_S = (\frac{x_S}{c_S}) dy$ for all S for which $t \in S$, then these changes exactly match up.

Ideally, we would follow exactly this pattern. However, notice that we start from $x^{(0)} = 0$, so all increases would be 0. Therefore, let $\eta > 0$ be very small and set

$$dx_S = \frac{1}{c_S}(x_S + \eta) dy \ .$$

This is a differential equation. We try a solution of the form $x_S = C_1 \mathrm{e}^{C_2 y} + C_3$. Then we have $\frac{dx_S}{dy} = C_2(x_S - C_3)$, so $C_3 = -\eta$, $C_1 = x_S^{(t-1)} + \eta$, $C_2 = \frac{1}{c_S}$. This way

$$x_S^{(t)} + \eta = \mathrm{e}^{\frac{1}{c_S} y_t} \left(x_S^{(t-1)} + \eta \right) \ ,$$

where y_t is the smallest value such that $x^{(t)}$ is a feasible solution to the first t constraints of the primal LP.

3 Algorithm

Let us now use the algorithmic approach above to design an algorithm for fractional online set cover.

For our algorithm, we set $\eta = \frac{1}{n}$ and initialize all $x_S = 0$. In the t-th step, when element t arrives, we introduce the primal constraint $\sum_{S:t\in S} x_S \geq 1$ and a dual variable y_t. We initialize $y_t = 0$ and update it as follows. For each S with $t \in S$ increase x_S from $x_S^{(t-1)}$ to $x_S^{(t)}$ by

$$x_S^{(t)} + \eta = \mathrm{e}^{\frac{1}{c_S} y_t} \left(x_S^{(t-1)} + \eta \right) \quad ,$$

where y_t is the smallest value such that $x^{(t)}$ is a feasible solution.

Theorem 3.2. *The algorithm is $O(\log n)$-competitive for fractional online set cover.*

Proof. We will verify the conditions of Lemma 3.1 with $\beta = 2$ and $\gamma = \ln(n+1)$.

We start by property (a). Consider the t-th step; element t arrives in this step. We have to relate $P^{(t)} - P^{(t-1)} = \sum_S c_S (x_S^{(t)} - x_S^{(t-1)})$ to y_t. For every set S such that $t \in S$, we have

$$x_S^{(t)} + \eta = \mathrm{e}^{\frac{1}{c_S} y_t} \left(x_S^{(t-1)} + \eta \right) \quad ,$$

and therefore

$$x_S^{(t-1)} + \eta = \mathrm{e}^{-\frac{1}{c_S} y_t} \left(x_S^{(t)} + \eta \right) \quad .$$

This lets us write the increase of x_S as follows

$$x_S^{(t)} - x_S^{(t-1)} = \left(x_S^{(t)} + \eta \right) - \mathrm{e}^{-\frac{1}{c_S} y_t} \left(x_S^{(t)} + \eta \right) = \left(1 - \mathrm{e}^{-\frac{1}{c_S} y_t} \right) \left(x_S^{(t)} + \eta \right) \leq \frac{1}{c_S} \left(x_S^{(t)} + \eta \right) y_t \quad .$$

This way, we can bound the primal increase by

$$P^{(t)} - P^{(t-1)} \leq \sum_{S:t\in S} c_S \frac{1}{c_S} \left(x_S^{(t)} + \eta \right) y_t = \sum_{S:t\in S} x_S^{(t)} y_t + \sum_{S:t\in S} \eta y_t \leq 2 y_t \quad ,$$

because $\sum_{S:t\in S} x_S^{(t)} = 1$ (otherwise we would have increased variables by too much) and $\sum_{S:t\in S} \eta \leq n\eta = 1$.

Now, we turn to property (b). Consider a fixed $S \in \mathcal{S}$. We will verify that the dual constraint for set S is fulfilled. By our algorithm if $t \in S$ then

$$y_t = c_S \ln(x_S^{(t)} + \eta) - c_S \ln(x_S^{(t-1)} + \eta) \quad ,$$

otherwise $x_S^{(t)} = x_S^{(t-1)}$ and so $c_S \ln(x_S^{(t)} + \eta) - c_S \ln(x_S^{(t-1)} + \eta) = 0$.

This lets us write the sum $\sum_{t\in S} y_t$ as

$$\sum_{t\in S} y_t = \sum_{t=1}^{m} \left(c_S \ln(x_S^{(t)} + \eta) - c_S \ln(x_S^{(t-1)} + \eta) \right) = c_S \ln \left(\frac{x_S^{(m)} + \eta}{x_S^{(0)} + \eta} \right) \quad .$$

Furthermore, $x_S^{(0)} \geq 0$ because variables are never negative and $x_S^{(m)} \leq 1$ because it does not make sense to increase variables beyond 1. So

$$\sum_{t:t\in S} y_t \leq c_S \ln \left(\frac{1+\eta}{\eta} \right) = c_S \ln(n+1) = \gamma c_S \quad .$$

□