

Algorithmic Game Theory

Summer Term 2023

Exercise Set 1

If you want to hand in your solutions for this problem set, please send them via email to anna.heuser@uni-bonn.de by Tuesday evening – make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.

If you would like to present one of the solutions in class, please also send an email to anna.heuser@uni-bonn.de containing the **task** which you would like to present and in **which of the tutorials** you would like to do so. Deadline for the email is Tuesday, 10:00 pm. Please note that the tasks will be allocated via a first-come-first-served procedure, so sending this email earlier than Tuesday evening is highly recommended.

Exercise 1: (6 Points)

Give an example for a symmetric network congestion game (strategies are s - t paths in a directed graph with the same s and t for all players) with monotonically increasing delay functions d_r such that there exist at least two pure Nash equilibria with different *social costs*. We define the social cost to be the sum of all players' costs $\sum_{i \in \mathcal{N}} c_i(S)$.

Exercise 2: (5+1 Points)

In a *weighted* congestion game, every player $i \in \mathcal{N}$ has an individual weight $w_i > 0$. The delay of a resource r now depends on the sum of the weights – instead of the number of players – of those players who are using r . For this purpose, we could redefine $n_r(S)$ to be $n_r(S) = \sum_{i: r \in S_i} w_i$ and consider delay functions $d_r: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Like in the unweighted case the cost of player i is defined as $c_i(S) = \sum_{r \in S_i} d_r(n_r(S))$.

- (a) Prove that weighted congestion games do not fulfil the *Finite-Improvement Property*, even having only two players, three resources and two strategies for each player.

Hint: Consider $\mathcal{N} = \{1, 2\}$, $w_1 = 1$, $w_2 = 2$, $R = \{a, b, c\}$, $\Sigma_1 = \{\{a\}, \{b, c\}\}$, $\Sigma_2 = \{\{b\}, \{a, c\}\}$. Choose delay functions such that

$$(\{a\}, \{b\}) \rightarrow (\{a\}, \{a, c\}) \rightarrow (\{b, c\}, \{a, c\}) \rightarrow (\{b, c\}, \{b\}) \rightarrow (\{a\}, \{b\})$$

is a sequence of best response improvement steps.

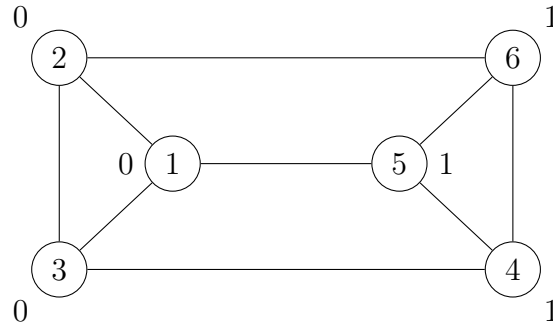
- (b) Use part (a) to show that a pure Nash equilibrium does not need to exist.

Exercise 3:

(1+3+2 Points)

In a *consensus game*, we are given an undirected graph $G = (V, E)$ with vertex set $V = \{1, \dots, n\}$. Each vertex $i \in V$ is a player and her action consists of choosing a bit $b_i \in \{0, 1\}$. Let $N(i) = \{j \in V \mid \{i, j\} \in E\}$ denote the set of neighbours of player i , i.e., all players j connected to i via an edge. Furthermore, let $\vec{b} = (b_1, \dots, b_n)$ be the vector of players' choices. The loss $D_i(\vec{b})$ for player i is the number of neighbours that she disagrees with, i.e.,

$$D_i(\vec{b}) = \sum_{j \in N(i)} |b_j - b_i|.$$



- Calculate the loss D_i of player 1 for the actions depicted in the graph above.
- Show that a consensus game represented as an undirected Graph G can also be modeled as a congestion game Γ . To this end, specify the tuple $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ and show that the loss D_i coincides with the cost c_i .
- Prove that in a congestion game modeling a consensus game with $|V| = n$ players all improvement sequences have length $O(n^2)$.