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Algorithmic Game Theory

Summer Term 2024

Tutorial Session - Week 1

You are supposed to work on these tasks in class together with your fellow students. Please find groups of 2 or 3 students!

If you do not know each other yet, each of you could start with a very quick introduction: What's your name? Do you study Computer Science or maybe something else (Maths, Economics,...)? Do you have any prior knowledge in Algorithmic Game Theory already or is this your first course in AGT?
Afterwards, you are supposed to discuss the exercises on this sheet. Note that you should see this also as a chance to talk about definitions, proof ideas and techniques used in the lecture in addition to only working out a formal solution for the tasks. If you do not know a definition or theorem by hard, feel free to open the lecture notes and have a look. Further, if you have any questions, I will drop by to discuss possible issues with you.

If there is some remaining time at the end of the tutorial, you can share your ideas on the tasks with the whole group.

Exercise 1:
In a *weighted* congestion game, every player $i \in \mathcal{N}$ has an individual weight $w_i > 0$. The delay of a resource r now depends on the sum of the weights – instead of the number of players – of those players who are using r. For this purpose, we could redefine $n_r(S)$ to be $n_r(S) = \sum_{i: r \in S_i} w_i$ and consider delay functions $d_r \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$. Like in the unweighted case the cost of player i is defined as $c_i(S) = \sum_{r \in S_i} d_r(n_r(S))$.

(a) Prove that weighted congestion games do not fulfil the *Finite-Improvement Property*, even having only two players, three resources and two strategies for each player.

Hint: Consider $\mathcal{N} = \{1, 2\}$, $w_1 = 1$, $w_2 = 2$, $R = \{a, b, c\}$, $\Sigma_1 = \{\{a\}, \{b, c\}\}$, $\Sigma_2 = \{\{b\}, \{a, c\}\}$. Choose delay functions such that

$$(\{a\}, \{b\}) \to (\{a\}, \{a, c\}) \to (\{b, c\}, \{a, c\}) \to (\{b, c\}, \{b\}) \to (\{a\}, \{b\})$$

is a sequence of best response improvement steps.

(b) Use part (a) to show that a pure Nash equilibrium does not need to exist.

Exercise 2:
Consider congestion games with a constant number of players. Show that the length of every sequence of improvement steps is bounded polynomially in the number of player strategies.