Exercise 1:

A connection game is a congestion game with \( n \) agents and an undirected graph \( G = (V, E) \). Every agent \( i \) is associated with a subset of vertices \( V_i \subseteq V \). The set of strategies \( \Sigma_i \) consists of all connected, acyclic subgraphs \( G'_i \) with \( V'_i = V_i \) and \( E'_i \subseteq (E \cap (V_i \times V_i)) \), for every player \( i \). Every edge \( e \) is assigned a delay function \( d_e(n_e) : \{1, ..., n\} \rightarrow \mathbb{Z} \), where \( n_e \) is the number of agents \( i \) selecting a subgraph \( G'_i \) with \( e \in E'_i \).

Exercise 1: (2+5 Points)

A connection game is a congestion game with \( n \) agents and an undirected graph \( G = (V, E) \). Every agent \( i \) is associated with a subset of vertices \( V_i \subseteq V \). The set of strategies \( \Sigma_i \) consists of all connected, acyclic subgraphs \( G'_i \) with \( V'_i = V_i \) and \( E'_i \subseteq (E \cap (V_i \times V_i)) \), for every player \( i \). Every edge \( e \) is assigned a delay function \( d_e(n_e) : \{1, ..., n\} \rightarrow \mathbb{Z} \), where \( n_e \) is the number of agents \( i \) selecting a subgraph \( G'_i \) with \( e \in E'_i \).

a) Consider the above instance of a connection game with two players. The vertices in \( V_1 \) are indicated in orange, while the vertices in \( V_2 \) are marked in blue.

Let the initial strategy of player 1 be given by the subgraph \( G'_1 \) with edges \( E'_1 = \\{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}\} \).

Player 2 chooses subgraph \( G'_2 \) with edges \( E'_2 = \\{\{v_1, v_5\}, \{v_2, v_5\}, \{v_4, v_5\}\} \) as his strategy.

Perform best-response improvement steps until a pure Nash equilibrium is reached. Player 1 should deviate first.
b) Prove: Every sequence of best-response improvement steps in a connection game converges in $O(n^2 \cdot |E| \cdot |V|)$ many steps.

Hint: You can use the following property without proving it. Let $G'$ be the strategy of agent $i$ in state $S$, and let $G''$ be a best response of $i$ for $S_{-i}$. Then, there exists a transforming sequence from $G'$ to $G''$, where in every step, one edge $e' \in (E' \setminus E'')$ is exchanged by an edge $e'' \in (E'' \setminus E')$. For each step, the resulting graph is a feasible strategy for agent $i$. In particular, the delay is (weakly) reduced in every step.

Exercise 2:  
(1+3+2 Points)

In a consensus game, we are given an undirected graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$. Each vertex $i \in V$ is a player and her action consists of choosing a bit $b_i \in \{0, 1\}$. Let $N(i) = \{j \in V \mid \{i, j\} \in E\}$ denote the set of neighbors of player $i$, i.e., all players $j$ connected to $i$ via an edge. Furthermore, let $\vec{b} = (b_1, \ldots, b_n)$ be the vector of players’ choices.

The loss $D_i(\vec{b})$ for player $i$ is the number of neighbors that she disagrees with, i.e.,

$$D_i(\vec{b}) = \sum_{j \in N(i)} |b_j - b_i|.$$ 

a) Calculate the loss $D_i$ of player 1 for the actions depicted in the graph above.

b) Show that a consensus game represented as an undirected Graph $G$ can also be modeled as a congestion game $\Gamma$. To this end, specify the tuple $\Gamma = (N, R, (\Sigma_i)_{i \in N}, (d_r)_{r \in R})$ and show that the loss $D_i$ coincides with the cost $c_i$.

c) Prove that in a congestion game modeling a consensus game with $|V| = n$ players all improvement sequences have length $O(n^2)$. 