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Due date: June 11, 2024 at 10:00 p.m.

Algorithmic Game Theory

Summer Term 2024

Exercise Set 5

If you want to hand in your solutions for this problem set, please send them via email to anna.heuser@uni-bonn.de *by Tuesday evening – make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.*

If you would like to present one of the solutions in class, please also send an email to anna.heuser@uni-bonn.de *containing the* ***task*** *which you would like to present and in* ***which of the tutorials*** *you would like to do so. Deadline for the email is Tuesday, 10:00 pm. Please note that the tasks will be alloecated via a first-come-first-served procedure, so sending this email earlier than Tuesday evening is highly recommended.*

Exercise 1: (3+1 Points)

A billionaire considers selling tours to the moon. The cost of building a rocket is C. Let $N = \{1, \ldots, n\}$ be the set of people who initially have declared an interest in the trip. The billionaire wishes to design a mechanism that will recover his cost but does not have information about the private valuation the bidders have for joining the trip. Therefore, he runs the following auction given as pseudocode:

- All bidders $i \in N$ simultaneously submit their bids $b_i \geq 0$.
- $S \leftarrow N$
- While $S \neq \emptyset$ do
  - $S' \leftarrow \{i \in S \mid b_i \geq \frac{C}{|S|}\}$
  - If $S' = S$, then allocate a seat for each $i \in S$ and no seat for each $i \in N \setminus S$. All bidders $i \in S$ have to pay $\frac{C}{|S|}$. The rest of the bidders $i \in N \setminus S$ has to pay nothing. Return.
  - Otherwise, $S \leftarrow S'$
- Do not allocate any seat and charge no payments at all. Return.

Show that the described mechanism is truthful.

Exercise 2: (5 Points)
Recall the *Greedy-by-Value* and *Greedy-by-Sqrt-Value-Density* algorithms for single-minded CAs of lecture 12. Let us analyse another greedy algorithm that looks as follows.

Greedy-by-Value-Density

- Re-order the bids such that $\frac{b_1^*}{|S_1^*|} \geq \frac{b_2^*}{|S_2^*|} \geq \cdots \geq \frac{b_n^*}{|S_n^*|}$.
- Initialize the set of winning bidders to $W = \emptyset$.
- For $i = 1$ to n do: If $S_i^* \cap \bigcup_{j \in W} S_j^* = \emptyset$, then $W = W \cup \{i\}$.

Let $d = \max_{i \in \mathcal{N}} |S_i^*|$. Show that the given algorithm yields a d-approximation.

Exercise 3: (4 Points)
As seen in lecture 13, let $f \colon V \to X$ be a function that maximizes declared welfare, i.e., $f(b) \in \arg\max_{x \in X} \sum_i b_i(x)$ for all $b \in V$. For each i, let h_i be an arbitrary function $b_{-i} \mapsto h_i(b_{-i})$ which does not depend on b_i. We define a mechanism $\mathcal{M} = (f, p)$ by setting

$$p_i(b) = h_i(b_{-i}) - \sum_{j \neq i} b_j(f(b)) \ .$$

Prove that $\mathcal{M}$ is a truthful mechanism.

Exercise 4: (4+2 Points)
Consider the following *Procurement Auction*. It's being attempted to buy a certain item. There are n vendors who are able to manufacture the wanted item. Vendor i incurs a cost of c_i for crafting the item. Now, the vendors are asked to state their costs for crafting the item and a vendor with lowest cost shall be chosen. The latter potentially gets a payment for it. The stated problem can be formalized by the general model of the lecture: Each vendor i is interpreted as a bidder who has negative valuation v_i, if he/she is chosen to craft the item, that is, $v_i(x) = -c_i$, if i is chosen in x.

(a) The results of the lecture concerning VCG are applicable in this situation. Make use of them in order to state a truthful mechanism.

(b) Use your results from the previous exercise to make the mechanism individually rational.

Exercise 5: (3+2 Points)
We consider a single-item auction via a mechanism which follows the spirit from Lecture 14, Section 2: All bidders submit their bids b_i. Fix a price of p (may depend on b) for the item. Approach bidders in order $1, \ldots, n$. As we consider bidder i: if the item is not allocated yet, assign the item for a price of p if $b_i - p \geq 0$.

(a) If $b = v$, show that the social welfare obtained by this auction is at least

$$\max_i v_i \mathbb{1}_{\text{item not allocated}} + p \left(\mathbb{1}_{\text{item allocated}} - \mathbb{1}_{\text{item not allocated}} \right) \ .$$

(b) Use your result from (a) to set a price obtaining a social welfare of at least $\frac{1}{2} \max_i v_i$ if $b = v$.