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Due date: May 18, 2026 at 8:00 p.m.

Algorithmic Game Theory

Summer Term 2026

Exercise Set 4

If you would like to submit your solutions for this problem set, please send them via email to aheuser1@uni-bonn.de by Monday evening. Submitting solutions in groups is also possible.

If you would like to present one of your solutions in class, please use the following link to book a presentation slot by Monday evening:
https://terminplaner6.dfn.de/b/46b3ae315324d5cf7c5f419a8739c8ec-1735102
A short meeting to discuss your solution is mandatory before presenting it in class. To book a time slot for this meeting, please use the following link by Monday evening as well:
https://terminplaner6.dfn.de/b/0f5e8ce957693f056cf215f62c33fbf6-1735091

Exercise 1:
Consider the following regret-minimization-algorithm.

Greedy

- Set $p_1^1 = 1$ and $p_j^1 = 0$ for all $j \neq 1$.
- In each round $t = 1, \dots, T$:
  Let $L_{min}^t = \min_{i \in N} L_i^t$ for $L_i^t = \sum_{t' \leq t} \ell_i^{(t')}$ and $S^t = \{i \in N \mid L_i^t = L_{min}^t\}$.
  Set $p_i^{t+1} = 1$ for $i = \min S^t$ and $p_j^{t+1} = 0$ otherwise.

You can assume that $\ell_i^{(t)} \in \{0, 1\}$ for all i and t.

(a) Show that the costs of Greedy are at most $N \cdot L_{min}^T + (N - 1)$.

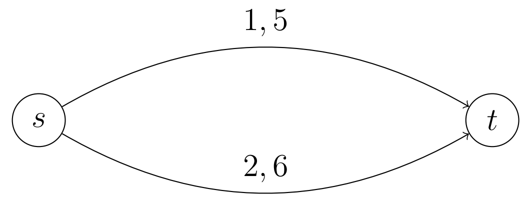
(b) State a scheme for an example such that the stated upper bound of (a) is tight for an infinite number of values T.

Exercise 2:
We consider the Multiplicative-Weights Algorithm with a slightly modified update rule. Instead of using $w_i^{(t+1)} = w_i^{(t)} \cdot (1 - \eta)^{\ell_i^{(t)}}$, we now use $w_i^{(t+1)} = w_i^{(t)} \cdot \left(1 - \eta \cdot \ell_i^{(t)}\right)$. Prove a statement as in Proposition 7.7. for this modified update rule.

Exercise 3:
Referring to the Price of Anarchy from Lecture 8, we introduced a more optimistic point of view called the *Price of Stability* in Lecture 9. For an equilibrium concept Eq, it is defined as

$$PoS_{\mathsf{Eq}} = \frac{\min_{p \in \mathsf{Eq}} SC(p)}{\min_{s \in S} SC(s)} \ .$$

Consider the following symmetric network congestion game with two players:



(a) What is the Price of Anarchy and the Price of Stability of pure Nash equilibria?

(b) What is the Price of Anarchy and the Price of Stability of mixed Nash equilibria?

Hint: First of all, determine all mixed Nash equilibria. You might start with a sentence like "Let σ be a mixed Nash equilibrium with $\sigma_1 = (p_1, 1-p_1)$, $\sigma_2 = (p_2, 1-p_2)$" and subsequently derive properties of p_1 and p_2.

Exercise 4:
Consider a (λ, μ)-smooth game with N players and let $s^{(1)}, \ldots, s^{(T)}$ be a sequence of states such that the external regret of every player is at most $R^{(T)}$. Moreover, let s^* denote a state that minimizes the social cost. We want to upper bound the average social cost of the sequence of states. To this end, show the following bound

$$\frac{1}{T}\sum_{t=1}^{T} SC(s^{(t)}) \leq \frac{N \cdot R^{(T)}}{(1-\mu)T} + \frac{\lambda}{1-\mu} SC(s^*)\,.$$

Hint: In this setting, the external regret for player i is the difference between the cost they have incurred and the cost they would have incurred with the best fixed strategy in hindsight.

Exercise 5:
A *fair cost-sharing game* is a congestion game such that for all resources $r \in R$ the delay function can be modeled as $d_r(x) = c_r/x$ for a constant c_r.
Show that fair cost sharing games with n players are $(n, 0)$-smooth.