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November 13, 2018
Due: lecture on November 20, 2018

Algorithms and Uncertainty

Winter Semester 2018/19

Exercise Set 4

Exercise 1: (4 Points)
We extend the problem of opening boxes from Lecture 8. Now, we may keep up to ℓ prizes. Show that the adaptivity gap is still at most 8.

Exercise 2: (3+3 Points)
In the Stochastic Knapsack problem, like in the non-stochastic variant, items of values v_i and sizes s_i are packed into a knapsack of a fixed capacity c; the goal is to maximize the sum of values of packed items. In the stochastic variant, the sizes are random. We get to know the size of an item only *after* we pack it into the knapsack. If it exceeds the knapsack capacity, the item is removed from the knapsack and no further items can be packed. Again, optimal policies are adaptive, meaning that their choices depend on the sizes of already packed items.

(a) Show that every policy corresponds to a solution of the following linear program

$$\begin{aligned} \text{maximize} \quad & \sum_i v_i x_i \\ \text{subject to} \quad & \sum_i \mathbf{E}[\min\{s_i, c\}] x_i \leq 2c \\ & 0 \leq x_i \leq 1 \qquad\qquad \text{for all } i \end{aligned}$$

(b) For simplicity, we assume that $s_i \leq \frac{1}{2}c$ for all i with probability 1. Now, we use an optimal LP solution x^* to determine a randomized non-adaptive policy as follows: Pack item i into the knapsack with probability $\frac{1}{4}x_i^*$. Show that this policy achieves an expected reward of at least $\frac{1}{8}\sum_i v_i x_i^*$.

Exercise 3: (3+3 Points)
In this exercise, we improve the approximation result for Stochastic Vertex Cover. Our algorithm again solves the LP relaxation. Then, it draws a threshold θ uniformly from $[0, \frac{1}{2}]$. In the first stage, it picks all vertices for which $x_v \geq \theta$. In the second stage in scenario E it picks all vertices for which $x_v < \theta$ but $x_v + y_{E,v} \geq \theta$.

(a) Show that the vertex cover is always feasible (that is, with probability 1).

(b) Show that the expected cost of the vertex cover is at most 2-times the cost of the optimal LP solution.

Exercise 4 on the next page.

Exercise 4: (4 Points)

We consider Stochastic Vertex Cover but now there are three phases. The edge set is determined as follows. First draw a number $K \in \{1, \ldots, k\}$ from the probability distribution $(p_i)_{i \in \{1,\ldots,k\}}$. Then, we draw the set E from the probability distribution $(q_E^K)_E$. The the first phase, E is completely unknown. In the second phase, K is known. Finally, in the third phase, E is known. Vertices can be added at cost $(c_v^{\mathrm{I}})_{v \in V}$, $(c_v^{\mathrm{II}})_{v \in V}$, and $(c_v^{\mathrm{III}})_{v \in V}$ respectively. Write an LP relaxation as in the lecture and use an optimal solution to give a 6-approximation to the optimal policy.