MA-INF 1203 Discrete and Computational Geometry

Wintersemester 2019/20
Assignment 1

To be discussed: **15/16**. October 2019

1 Asymptotic notations

i) In each of the following situations, determine whether $f(n) = O(g(n))$, or $f(n) = \Omega(g(n))$, or both (i.e. $f(n) = \Theta(g(n))$). We denote with $\log n$ the logarithm of base 2.

$f(n)$	$g(n)$
$100 \log n$	$\log(n^3)$
$n^{1.001}$	$n \log^{100} n$
$4^{\log n}$	n^2
$\sum_{i=1}^{n} \frac{1}{i}$	$\ln n$
$\sum_{i=1}^{n} \frac{1}{i}$	$\log n$
2^n	2^{n+5}
2^n	2^{2n}
$\binom{n}{5}$	n^5
$n!$	2^n
$\sum_{i=1}^{c} n^i$	n^c

ii) In each of the above situations determine whether $f(n) = o(g(n))$, or $f(n) = \omega(g(n))$, or none.

2 Probability Theory

Suppose that you play a simplified game of darts, in which the board is a square of side-length ℓ as depicted in Figure 1. Let $D_0, \dots D_5$ be the concentric disks in Figure 1, with radii $r_0, \dots, r_5$ respectively. The rules are as follows: hitting D_0 scores 100 points, hitting $D_1 \setminus D_0$ scores 50 points, hitting $D_2 \setminus D_1$ scores 40 points, hitting $D_3 \setminus D_2$ scores 30 points, hitting $D_4 \setminus D_3$ scores 20 points, and hitting $D_5 \setminus D_4$ scores 10 points.

Let $r_0 = 2$, $r_1 = 3$, $r_2 = 4$, $r_3 = 5$, $r_4 = 6$, $r_5 = 7$ and $\ell = 16$. Assume that a dart hits a point on the board uniformly at random and any dart hits the board with probability 1.

i) What is the probability of scoring at least 30 points?

ii) What is the probability of scoring exactly 30 points?

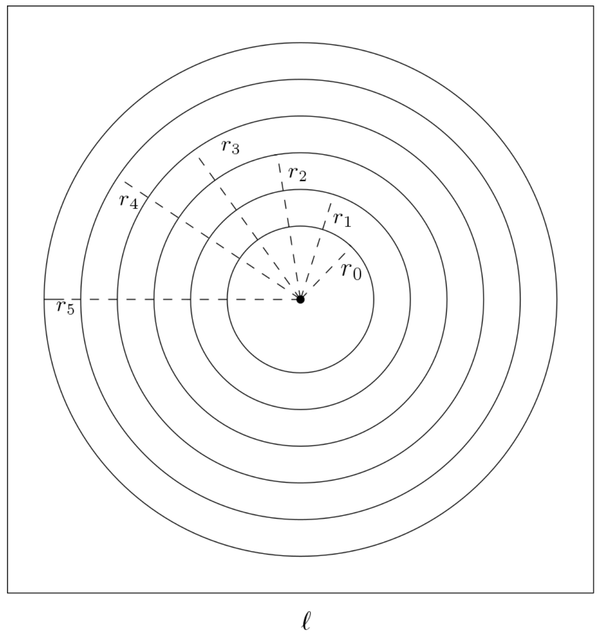

Figure 1

iii) What is the expected score?

Now consider a different set of rules: hitting D_0 in at least 1 out of 5 independent attempts gives a win, while otherwise the game is lost. Once again assume that each dart hits a point on the board uniformly at random and each dart hits the board with probability 1. The 5 independent attempts correspond to 5 independent shots.

iv) What is the probability of winning the game?

3 Recurrences

Note that $\log n$ denotes the logarithm of base 2.

i) Let $T(n) = 2T(\lfloor n/2 \rfloor) + n$, and $T(1) = 1$. Use mathematical induction to prove that $T(n) = O(n \log n)$. Please state explicitly the base case, the induction hypothesis and the inductive step.

ii) Solve the recurrence $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n$, where $T(1) = 1$.

iii) Solve the recurrence

$$T(n) \leq \begin{cases} 1 & \text{if } n = 1 \\ 1 + \sum_i T(n_i) & \text{if } n > 1, \end{cases}$$

where $\sum_i n_i = n$, and each $n_i \leq 2n/3$.